Model Theory and Differential Algebra

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Question: What is model theory?

Answer: Model theory is the study of *models*, structures which interpret formal languages.

Definition 1 *A signature σ is a quadruple $(\mathcal{C}, \mathcal{F}, \mathcal{R}, a)$ where $\mathcal{C}$, $\mathcal{F}$, and $\mathcal{R}$ are disjoint sets (called the* constant symbols, function symbols, *and* relation symbols, *respectively) and $a : \mathcal{F} \cup \mathcal{R} \to \mathbb{Z}_+$ is a function which assigns the* arity *of a function symbol or relation symbol.*

Definition 2 *If $\sigma = (\mathcal{C}, \mathcal{F}, \mathcal{R}, a)$ is a signature, then a σ-structure is a nonempty set M given together with an interpretation of σ. That is, for each $c \in \mathcal{C}$ one is given some $c^M \in M$. For each $f \in \mathcal{F}$ one is given $f^M : M^{a(f)} \to M$. For each $R \in \mathcal{R}$ one is given $R^M \subseteq M^{a(R)}$.*

In most cases under consideration here, σ will be the signature of differential rings. That is, $\mathcal{C} = \{0, 1\}$, $\mathcal{F} = \{+, \cdot, \partial\}$, $\mathcal{R} = \varnothing$, and $a(+) = a(\cdot) = 2$ while $a(\partial) = 1$. Our σ-structures will be differential rings and the symbols of σ will be interpreted in the usual way.

To each signature $\sigma = (\mathcal{C}, \mathcal{F}, \mathcal{R}, a)$ there is an associated (first-order) formal language built from the symbols in $\mathcal{C} \cup \mathcal{F} \cup \mathcal{R}$, a set of variable

names $\{x_i : i \in \mathbb{N}\}$, symbols for logical Boolean operations $\wedge, \vee, \to, \neg$, and quantification over elements $(\exists x_i)$ and $(\forall x_i)$.

Definition 3 *If $\sigma = (\mathcal{C}, \mathcal{F}, \mathcal{R}, a)$ is a signature, then the set of σ-terms is defined by the following recursion.*

- *c is term for any constant symbol $c \in \mathcal{C}$.*
- *x_i is a term for any natural number $i \in \mathbb{N}$.*
- *$f(t_1, \ldots, t_n)$ is a term if $f \in \mathcal{F}$ is a function symbol with $n = a(f)$ and $t_1, \ldots, t_n$ are all terms.*

Definition 4 *If σ is a signature, then the set of formulas of language associated to σ, $\mathcal{L}(\sigma)$, is defined by the following recursion.*

- *$t_1 = t_2$ is a formula if t_1 and t_2 are σ-terms.*
- *$R(t_1, \ldots, t_n)$ is a formula if $R \in \mathcal{R}$ is a relation symbol with $n = a(R)$ and $t_1, \ldots, t_n$ are all σ-terms.*

- $(\varphi \wedge \psi)$ *[read as "φ and ψ"] is a formula if φ and ψ are both formulas.*
- $\neg(\varphi)$ *[read as "not φ"] is a formula if φ is a formula.*
- $(\exists x_i)(\varphi)$ *[read as "There exists x_i such that φ."] is a formula if φ is a formula.*

If M is a σ-structure, then each formula in $\mathcal{L}(\sigma)$ has a natural interpretion in M.

If all the variables of the formula ψ are bound by a quantifier (such a formula is called a *sentence*), then M must decide the truth value of ψ. We write $M \models \psi$ [read "M models ψ"] if M interprets ψ as true.

If T is a set of sentences, then we write $M \models T$ iff $M \models \psi$ for every $\psi \in T$.

Definition 5 *The* theory of M, $\mathrm{Th}(M)$, *is the set of all σ-sentences which are true in M.*

If some of the variables of ψ are free, then ψ defines a subset of some power of M. If the free variables of ψ are among $x_1, \ldots, x_n$, then we write $\psi(M) := \{(a_1, \ldots, a_n) \in M^n : M \models \psi(a_1, \ldots, a_n)\}$ where $\psi(a_1, \ldots, a_n)$ denotes the result of substituting a_i for the variable x_i.

Example 1 *If σ is the signature of differential rings and R is a differential ring considered as a σ-structure in the natural way, and $\varphi := (\exists x_2)(\neq (x_2 \cdot \partial(x_1) = 1))$, then $\varphi(R) = \{a \in R : \partial(a) \in R^\times\}$.*

In Weil's approach to the foundations of algebraic geometry, a central role is played by the notion of a universal domain: an algebraically closed field into which every "small" field of the same characteristic may be embedded and for which any isomorphism between "small" subfields may be extended to an automorphism.

Question: Is there an analogous notion of universal domain in differential algebra?

For many natural theories there are no universal domains.

Abraham Robinson arrived at a positive answer to this question by finding the *model completion* of the theory of differential fields of characteristic zero.

Definition 6 *The theory T' is a model companion of the theory T if*

- *T and T' are co-theories: every model of T may be extended to a model of T' and* vice versa *and*
- *every model of T' is existentially closed: if $M, N \models T'$, $\varphi(\vec{x}, y_1, \ldots, y_n)$ is a quantifier free formula, $a \in M^n$, and $N \models (\exists \vec{x})\varphi(\vec{x}, a)$, then $M \models (\exists \vec{x})\varphi(\vec{x}, a)$.*

If relative to T' every formula is equivalent to a quantifier free formula, then T' is called a model completion *of T.*

If T has a model companion, then it has only one.

Example 2

- *The theory of algebraically closed fields is the model completion of the theory of fields.*
- *The theory of real closed fields is the model companion of the theory of formally real fields. Considered with the signature* $(\{0, 1\}, \{+, \cdot\}, \{<\})$ *it is the model completion of the theory of ordered fields.*

Theorem 7 *The model completion of the theory of differential fields of characteristic zero is the theory of differentially closed fields of characteristic zero,* DCF_0.

The fact that DCF_0 eliminates quantifiers also takes a geometric form.

Proposition 8 *If $K \models \mathrm{DCF}_0$, $X \subseteq K^n$ is Kolchin-constructible, and $f : K^n \to K^m$ is a differential rational function, then $f(X) \subseteq K^m$ is also Kolchin-constructible.*

There are a few reasonable ways to axiomatize DCF_0.

Definition 9 *A differential field of characteristic zero K is differentially closed if for each pair $f, g \in K\{x\}$ of differential polynomials with f irreducible and g simpler than f, there is some $a \in K$ with $f(a) = 0$ and $g(a) \neq 0$.*

Ehud Hrusovski provided geometric axioms. Before we can state the geometric axioms, we need to recall the definition of jet spaces.

Definition 10 *If (K, ∂) is a differential field of characteristic zero, X is a scheme over K, and $n \in \mathbb{N}$ is a natural number, then the n-th jet space of X is the scheme $\nabla_n X$ which represents the functor $K - \partial - Alg \to \mathbf{S}ets$ given by $(R, \partial) \mapsto X_{R[\epsilon]/(\epsilon^{n+1})}(R[\epsilon]/(\epsilon^{n+1}))$ where X is made into a scheme over $R[\epsilon]/(\epsilon^{n+1})$ via the map $x \mapsto \sum_{i=0}^{n} \frac{1}{n!}\partial^n(x)$.*

Concretely, if $X = \mathrm{Spec} K[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, then

$\nabla_1 X = \mathrm{Spec} K[x_1, \ldots, x_n; x_1', \ldots, x_n']/(f_1, \ldots, f_m, \sum_{j=1}^{n} \frac{\partial f_1}{\partial x_j} x_j' - f_1^{\partial}, \ldots, \sum_{j=1}^{n} \frac{\partial f_m}{\partial x_j} x_j' - f_m^{\partial})$ where g^{∂} denotes the result of applying ∂ to the coefficients of g.

The reduction map $R[\epsilon]/(\epsilon^{n+1}) \to R[\epsilon]/(\epsilon^{m+1})$ corresponds to a projection $\pi : \nabla_n X \to \nabla_m X$.

Proposition 11 *A differential field of characteristic zero K is differentially closed if and only if for any irreducible affine variety X over K and Zariski constructible set $W \subseteq \nabla_1 X$ with $\pi \restriction_W : W \to X$ dominant, there is some point $a \in X(K)$ with $(a, \partial a) \in W(K)$.*

The theory of differentially closed fields of characteristic zero is a *totally transcendental* theory.

Many deep theorems have been proven about general totally transcendental theories, but for all practical purposes, the theory of differentially closed fields is the only known mathematically significant theory to which the deeper parts of the general theory applies.

Definition 12 *Let T be a theory, $M \models T$ a model of T and $A \subseteq M$ a subset. A* prime model *of T over A is a model $P \models T$ with $A \subseteq P \subseteq M$ having the property that if $\iota : A \hookrightarrow N$ is an embedding A into any other model $N \models T$, then ι extends to an embedding of P into N.*

Theorem 13 (Shelah) *If T is a totally transcendental theory, then for any model $M \models T$ and subset $A \subseteq M$ there is prime model over A. Moreover, the prime model is unique up to isomorphism over A.*

Corollary 14 *If K is a differential field of characteristic zero, then there is a differentially closed differential field extension K^{dif}/K, called the differential closure of K, which embeds over K into any differentially closed extension of K and which is unique up to K-isomorphism.*

The theory of algebraically closed fields is also totally transcendental and the prime model over a field K is its algebraic closure K^{alg}. The algebraic closure is also *minimal*. That is, if $K \subseteq L \subseteq K^{alg}$ with L algebraically closed, then $L = K^{alg}$.

The differential closure does not share this property.

Theorem 15 (Kolchin, Rosenlicht, Shelah) *If K is a differential closure of $\mathbb{Q}$, then there are $\aleph_0$ differentially closed subfields of K.*

The nonminimality of differential closures results from the existence of *trivial* differential equations. In this context, *trivial* does not mean *easy* or *unimportant*. Rather, it means that an associated combinatorial geometry is degenerate.

Definition 16 *A combinatorial pregeometry is a set S given together with a closure operator* $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$ *satisfying universally*

- $X \subseteq \mathrm{cl}(X)$
- $X \subseteq Y \Rightarrow \mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$
- $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$
- *if* $a \in \mathrm{cl}(X \cup \{b\}) \setminus \mathrm{cl}(X)$*, then* $b \in \mathrm{cl}(X \cup \{a\})$.
- *if* $a \in \mathrm{cl}(X)$*, then there is some finite* $X_0 \subseteq X$ *such that* $a \in \mathrm{cl}(X_0)$.

If (S, cl) *satisfies* $\mathrm{cl}(\varnothing) = \varnothing$ *and* $\mathrm{cl}(\{x\}) = \{x\}$*, then we say that* (S, cl) *is a combinatorial geometry.*

Example 3

- *If S is any set and* $\mathrm{cl}(X) := X$*, then* (S, cl) *is a combinatorial geometry.*
- *If S is a vector space over a field* K *and* $\mathrm{cl}(X) :=$ *the* K*-span of* X*, then* (S, cl) *is a combinatorial pregeometry.*
- *If S is an algebraically closed field and* $\mathrm{cl}(A)$ *is the algebraic closure of the field generated by* A*, then* (S, cl) *is a combinatorial pregeometry.*

Definition 17 *The pregeometry* (S, cl) *is* trivial *if for any* $X \in \mathcal{P}(S)$ *one has* $\mathrm{cl}(X) = \bigcup_{x \in X} \mathrm{cl}(\{x\})$.

Definition 18 *If* (S, cl) *is a pregeometry, then a set* $X \subseteq S$ *is* independent *if for any* $x \in X$ *one has* $x \notin \mathrm{cl}(X \setminus \{x\})$.

Proposition 19 *If* (S, cl) *is a pregeometry,* $A \subseteq S$*, and* $X, Y \subseteq A$ *are two maximal independent subsets of* A*, then* $\|X\| = \|Y\|$*. We define* $\dim(A) := \|X\|$.

Definition 20 *A combinatorial pregeometry* (S, cl) *is* locally modular *if whenever* $X, Y \subseteq S$ *and* $\dim(\mathrm{cl}(X) \cap \mathrm{cl}(Y)) > 0$ *we have*
$\dim(\mathrm{cl}(X) \cap \mathrm{cl}(Y)) + \dim(\mathrm{cl}(X \cup Y)) = \dim(\mathrm{cl}(X)) + \dim(\mathrm{cl}(Y))$.

Definition 21 *Let* M *be a* σ*-structure for some signature* σ*. Let* $\psi(x_1, \ldots, x_n)$ *be some* σ*-formula with free variables among* $x_1, \ldots, x_n$.

We say that the set $D := \psi(M)$ *is* strongly minimal *if* $\psi(M)$ *is infinite and for any* $N \succeq M$ *and any formula* $\phi(x_1, \ldots, x_n) \in \mathcal{L}_N(\sigma)$ *either* $\psi(N) \cap \phi(N)$ *is finite or* $\psi(N) \cap (\neg\phi)(N)$ *is finite.*

Definition 22 *Let* M *be a* σ*-structure for some signature* σ*. Let* $A \subseteq M$*. We say that* $a \in M$ *is* model theoretically algebraic *over* A *if there is a formula* $\psi(x) \in \mathcal{L}_A(\sigma)$ *such that* $M \models \psi(a)$ *but* $\psi(M)$ *is finite. We denote by* $\mathrm{acl}(A)$ *the set of all elements of* M *which are algebraic over* M*.*

Example 4 *If* K *is a differentially closed field and* $A \subseteq K$*, then* $\mathrm{acl}(A) = Q\langle A\rangle^{alg}$.

Proposition 23 *Let* D *be a strongly minimal set. Define* $\mathrm{cl} : \mathcal{P}(D) \to \mathcal{P}(D)$ *by* $X \mapsto \mathrm{acl}(X) \cap D$*. Then* (D, cl) *is a combinatorial pregeometry.*

Remark Generally, one gets very little information from the pregeometry

of a strongly minimal set unless the ambient model is fairly large.

Conjecture 24 (Zilber) *If* D *is a strongly minimal set whose associated pregeometry is not locally modular, then* D *interprets an algebraically closed field.*

Theorem 25 (Hrushovski) *Zilber's conjecture is false.*

Theorem 26 (Hrushovski-Zilber) *Zilber's conjecture holds for* Zariski geometries, *strongly minimal sets satisfying certain topological and smoothness properties.*

Theorem 27 (Hrushovski-Sokolović) *Every strongly minimal set in a differentially closed field is a Zariski geometry after finitely many points are removed. Hence, Zilber's conjecture is true for strongly minimal sets in differentially closed fields. In fact, if* D *is a non-locally modular* *strongly minimal set defined in some differentially closed field K, then there is a differential rational function f for which $f(D) \cap K^{\partial}$ is finite, where $K^{\partial} := \{c \in K : \partial c = 0\}$.*

Theorem 27 is instrumental in the analysis of the structure of differential algebraic groups.

Theorem 28 (Hrushovski-Pillay) *Suppose that $D_1, \ldots, D_n$ are locally modular strongly minimal sets, G is a definable group, and $G \subseteq \mathrm{acl}(D_1 \cup \cdots \cup D_n)$. Then every definable subset of any power of G is a finite Boolean combination of cosets of definable subgroups.*

We call a group satisfying the conclusion of Theorem 28 *weakly normal*.

Theorem 29 (Manin, Buium) *If A is an abelian variety of dimension g defined over a differentially closed field of characteristic zero K, then*

there is a surjective differential rational homomorphism $\mu : A(K) \to \mathbb{G}_a(K)^g$.

The kernel of μ is denoted by $A^{\sharp}$ and is called the *Manin kernel* of A.

Theorem 30 (Buium, Hrushovski) *If A is an abelian variety defined over a differentially closed field K and A admits no non-zero algebraic homomorphisms to abelian varieties defined over K^{∂}, then $A(K)$ is weakly normal.*

Corollary 31 (Function field Manin-Mumford conjecture) *If A if an abelian variety defined over a field K of characteristic zero, A does not admit any nontrivial algebraic homomorphisms to abelian varieties defined over $\mathbb{Q}^{alg}$, and $X \subseteq A$ is an irreducible variety for which $X(K) \cap A(K)_{\mathrm{tor}}$ is Zariski dense, then X is a translate of an algebraic subgroup of A.*

The function field Mordell-Lang conjecture follows from Theorem 30 together with a general result of Hrushovski on the structure of finite rank groups.

Definition 32 *If G is a differential algebraic group of finite rank, then the* socle *of G, $G^\flat$, is the maximal connected differential algebraic subgroup of G for which $G^\flat \subseteq \mathrm{acl}(D_1, \ldots, D_n)$ for some strongly minimal sets $D_1, \ldots, D_n$.*

Proposition 33 *Let G be a differential algebraic group of finite rank and $X \subseteq G$ an irreducible Kolchin constructible set. If the stabilizer of X is trivial, then X is contained in a coset of $G^\flat$.*

Remark The published proof of Proposition 33 deals with a more general class of groups and includes a *rigidity* hypothesis on $G^\flat$, but in the case of differential algebraic groups, this extra hypothesis is unnecessary.

Theorem 34 (Buium, Hrushovski) *If G is a semiabelian variety defined over a differentially closed field K, $X \subseteq G$ is an irreducible subvariety, $\Gamma \subseteq G(K)$ is a subgroup with $\dim_{\mathbb{Q}} \Gamma \otimes \mathbb{Q} < \infty$, and $X(K) \cap \Gamma$ is Zariski dense in X, then there is an algebraic subgroup $H \leq G$ of G, an algebraic group homomorphism $\psi : H \to H_0$ from H to an algebraic group H_0 defined over the constants K^∂, an algebraic variety $X_0 \subseteq H_0$ defined over K^∂ and a point $a \in G(K)$ such that $X = a + \psi^{-1} X_0$.*

Remark Of course, a stronger form of Theorem 34 (due to Faltings, Vojta, McQuillen, Bombieri, *et al*) in which one concludes that X is a translate of an algebraic subgroup of G holds.

As a consequence of the geometric axioms for differentially closed fields, Proposition 33, and intersection theory, Ehud Hrushovski and Anand Pillay derived explicit bounds on the number of generic points on subvarieties of semiabelian varieties.

Theorem 35 (Hrushovski, Pillay) *Let K be a finitely generated field extension of $\mathbb{Q}^{alg}$. Let G be a semiabelian variety defined over $\mathbb{Q}^{alg}$. Suppose that $X \subseteq G$ is an irreducible subvariety defined over $\mathbb{Q}^{alg}$ which cannot be expressed as $X_1 + X_2$ for some positive dimensional subvarieties of G. If $\Gamma < G(K)$ is a finitely generated group, then the number of points in $\Gamma \cap (X(K) \setminus X(\mathbb{Q}^{alg}))$ is finite and may be bounded by an explicit function of geometric data.*