Compact complex manifolds with the DOP and other properties.

Anand Pillay[*]
Dept. Mathematics
Univ. Illinois at Urbana-Champaign

Thomas Scanlon [†]
Dept. Mathematics
University of California at Berkeley

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Abstract

We point out that a certain complex compact manifold constructed by Lieberman has the dimensional order property, and has U-rank different from Morley rank. We also give a sufficient condition for a Kähler manifold to be totally degenerate (that is, to be an indiscernible set, in its canonical language) and point out that there are K3 surfaces which satisfy these conditions.

1 Introduction

This paper is concerned with model-theoretic properties of compact complex manifolds. A compact complex manifold X can be considered as a first order stucture by equipping it with predicates for the analytic subsets of $X, X \times X, X \times X \times X$ etc. As such $Th(X)$ is a theory of finite Morley

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rank. We can consider in the same way the family of *all* compact complex spaces as a many sorted structure $\mathcal{A}$ which will also be of finite Morley rank sort-by-sort.

If X happens to be a projective complex algebraic variety (or even a Moishezon space) then $Th(X)$ is bi-interpretable (with parameters) with the strongly minimal theory $Th(\mathbf{C}, +, \cdot)$, and the situation with regard to the kind of model-theoretic properties discussed in this paper is clear. We will show that a certain compact complex manifold X, produced many years ago by Lieberman [4], has various "bad" properties: $Th(X)$ has the DOP (Shelah's dimensional order property) and the generic type $p(x)$ of X has Morley rank different from U-rank. X will actually be a definable family of definable groups, and we also show that the generic fibre G provides a negative answer to a question in [12] (G will not be definably isomorphic to a group definable in an elementary extension of a Kähler manifold). Remarkably the example is similar to an example found in [7] of a finite Morley rank set defined in a differentially closed field, whose generic type has Morley rank different from U-rank. Our results follow fairly quickly from Campana's analysis [2] of Lieberman's example. In the next section we recall the example as well as Campana's observations, and conclude the main results.

Lieberman's example X is not in $\mathcal{C}$, where $\mathcal{C}$ denotes the class of compact complex manifolds which are bimeromorphic to Kähler manifolds. We do not at present know any compact complex space in $\mathcal{C}$ which has the DOP (or is even nonmultidimensional). The class $\mathcal{C}$ is considered to be more amenable to structure theorems, and attention is often restricted to this class. Spaces in $\mathcal{C}$ are also more manageable from the model-theoretic point of view as they essentially have a countable language and are thus saturated (see [9]). We conjecture that any trivial U-rank 1 type in $\mathcal{C}$ is the generic type of some strongly minimal ω-categorical set. In this connection we prove that a strongly minimal compact complex manifold (in $\mathcal{C}$) which is simply connected and has no automorphisms *is* ω-categorical; in fact it is an infinite set with no structure. Certain K3 surfaces furnish examples.

We use standard notation from stability theory and stable group theory.

The basic observation that compact complex manfolds have finite Morley rank is due to Zilber[15]. Further observations were made in [10] and the reader is referred to the latter paper for more background.

It is worth saying a little more at this stage regarding Lieberman and Campana's interest in the example X. Lieberman was interested in automorphism groups of compact complex manifolds. He observes that the connected component $\mathrm{Aut}^0(X)$ of $\mathrm{Aut}(X)$ is $\mathbf{C}$ and acts on X with a Zariski-dense orbit. This implies that the the action of $\mathrm{Aut}^0(X)$ on X is not definable in the structure $\mathcal{A}$ (otherwise the Zariski-dense orbit would be definable, and of dimension 1 showing that X has dimension 1, contradiction).

Campana was interested in producing strong counterexamples to possible generalizations of a result of Fischer and Forster [3]. Fischer and Forster proved that all but finitely many co-dimension 1 irreducible subvarieties of a compact complex space come from meromorphic functions. The issue was what about codimension bigger than 1. Campana finds infinitely many irreducible 1-dimensional subvarieties of X which are "inert", that is do not belong to nice infinite families. The Fischer-Forster result was used in [11] to show that if Y is a compact complex manifold of dimension at most 2 then its generic type has Morley rank equal to U-rank. So Campana's observation suggests that the example may have generic type with Morley rank different from U-rank. This turns out to be the case.

2 Main results.

We will give some details about the example for the benefit of the reader, although we will only really be making use of a part of Campana's observation (Fact 2.1 below).

Fix an elliptic curve E_0. We will write the group operation additively, so 0 denotes the identity of E_0. Let u be following automorphism of $E_0 \times E_0$: $u(a, b) = (a + b, a + 2b)$. Let τ be a complex number whose imaginary part is strictly positive. Let $\mathbf{Z} \times \mathbf{Z}$ act on the complex analytic manifold $E_0 \times E_0 \times \mathbf{C}$ as follows: $(m, n) \cdot ((a, b), s) = (u^n(a, b), s + m + n\tau)$. Let X be the quotient (that is the set of orbits). Then X has naturally the structure of a compact complex manifold. The canonical projection from $E_0 \times E_0 \times \mathbf{C}$ to $\mathbf{C}$ induces a holomorphic surjection q from X onto the elliptic curve E, where E is the quotient of $\mathbf{C}$ by the lattice L generated by 1 and τ. Let X_e denote $q^{-1}(e)$ and X_U denote $q^{-1}(U)$ for $e \in E$ and $U \subseteq E$.

X depends on E_0 and τ. We start with some elementary obervations about X and then we will state what Campana proved in the case that τ is sufficiently general.

I. q is locally trivial in the Euclidean topology. In fact for each $e \in E$ there

is a complex open neighbourhood U of e in E such that X_U is biholomorphic to $E_0 \times E_0 \times U$.
Explanation. Let $s \in \mathbf{C}$ be a representative of e. Let U be a small open neighbourhood of s in $\mathbf{C}$ biholomorphic (under quotienting by L) with an open neighbourhood of e in E. For s' in U, map the orbit under $\mathbf{Z}^2$ of $((a,b),s')$ to $((a,b),s'+L)$.

II. There is an analytic map S from $X \times_E X$ to X, such that for each $e \in E$, the restriction of S to $X_e \times X_e$ is precisely the group operation on $E_0 \times E_0$ (under any of the identifications of X_e with $E_0 \times E_0$ given in I).
Explanation. Fix $e \in E$, and $s \in \mathbf{C}$ a representative of e. Let f_s be the isomorphism of X_U with $E_0 \times E_0 \times U$ given as in Explanation of I. Let S_U on $X_U \times_U X_U$ be induced from f_s by the group operation on $E_0 \times E_0$. The only thing to see is that S does not depend on the choice of s. If $s' + L = s + L$ then $s' = s + m + n\tau$ for some m, n and $f_{s'}$ is the composition of f_s with (u^n, id). As u^n is an automorphism of $E_0 \times E_0$, S_U is unchanged. This shows that the S_U fit together to give analytic S as required.

By II, X has the analytic structure of a family of abelian varieties, each isomorphic to $E_0 \times E_0$. For each $n \geq 1$ let C_n be the 1-dimensional analytic subvariety of X such that for $e \in E$, $C_n \cap X_e$ is the subgroup of $E_0 \times E_0$ consisting of elements whose order divides n. In particular C_0 is the image of the 0-section of q.

Let u' be the lift of u to an element of $GL(2, \mathbf{C})$ and let λ be an eigenvalue of u'. Campana proves:

Fact 2.1 *Suppose that τ is sufficiently generic in the sense that $e^{2\pi i n\tau} \neq \lambda^m$ for all $m, n \in \mathbf{Z}$. Then every component of each C_n is a maximal irreducible subvariety of X. Moreover these are precisely the irreducible curves in X which project onto E under q.*

¿From now on we assume that τ is chosen as in Fact 2.1. All we will be using of Fact 2.1 is that C_0 is maximal irreducible. The model-theoretic analysis will yield a little more: X has no irreducible 2-dimensional analytic subvariety which projects onto E under q. We will be working in a saturated elementary extension of $\mathcal{A}$, which we call $\mathcal{A}'$. X', E' etc. denote the extensions of X, E etc. $\mathcal{A}'$ has finite Morley rank and we use the language of stability theory and stable groups.

Lemma 2.2 *Let e' be a generic point of E' over $\mathcal{A}$. Let $X'_{e'}$ be the fibre above e'. Then $X'_{e'}$ is a definably connected definable group with no proper infinite definable connected subgroups.*

Proof. Note first that any two definable connected subgroups of $E_0 \times E_0$ which have the same 2-torsion are the same. This tranfers to $X'_{e'}$ to show that any connected definable subgroup of $X'_{e'}$ is $acl(e')$-definable. Suppose G to be an infinite proper connected $acl(e')$-definable subgroup of $X'_{e'}$. Let a' be the generic point (in the model-theoretic sense) of G over $acl(e')$. Then $a' \in X'$ is the generic point of an irreducible subvariety Y say of X. As $q'(a') = e'$ is a generic point of E, $q(Y) = E$. As G is a subgroup it contains the 0 of $X'_{e'}$. It follows that Y contains C_0. Clearly Y is a proper irreducible subvariety of X of dimension 2, contradicting Fact 2.1

Lemma 2.3 *$X'_{e'}$ is orthogonal to $\mathbf{P}'_1$ (that is their generic types are orthogonal).*

Proof. By Lemma 2.2, $X'_{e'}$ is almost strongly minimal, so if it is nonorthogonal to $\mathbf{P}'_1$, some quotient of $X'_{e'}$ by a finite subgroup is definably isomorphic to an algebraic group H say (over $\mathbf{C}'$). Let H be definable over parameter $c \in \mathbf{C}'$, and write H as H_c. Note that H_c has no proper infinite definable subgroups. The fact that H_c is definably isomorphic to a quotient of $X'_{e'}$ by a finite subgroup, for some $e' \in E'$ is witnessed by a formula $\phi(x)$ (with parameters from $\mathcal{A}$). Clearly (by separation of parameters say), $\phi(x)$ is implied by $tp(c/\mathbf{P}_1)$. We consider now $\mathbf{P}_1$ in the basic language of fields (i.e. with predicates for the graph of addition and multiplication). As such $\mathbf{P}_1$ is saturated. There is a finite subset A of $\mathbf{P}_1$ such that $tp(c/A)$ implies $\phi(x)$. By saturation of $\mathbf{P}_1$, there is a tuple c_0 from $\mathbf{P}_1$ (that is in the standard model) with $tp(c_0/A) = tp(c/A)$. Then H_{c_0} has no proper infinite definable subgroups, but is definably isomorphic to $X_e = E_0 \times E_0$ for some $e \in E$, which is a contradiction.

Lemma 2.4 *$X'_{e'}$ is modular and strongly minimal.*

Proof. We have seen that $X'_{e'}$ is an almost strongly minimal group, which is orthogonal to $\mathbf{P}'_1$. By the Hrushovski-Zilber dichotomy theorem and its validity in $\mathcal{A}'$ (see [5] and [10]), $X'_{e'}$ is a modular group. As it has no proper infinite definable subgroups, $X'_{e'}$ must be strongly minimal.

Corollary 2.5 *X has no 2-dimensional irreducible subvariety Y which projects onto E under q. (So by Fact 2.1, the only irreducible subvarieties of X projecting onto E are the irreducible components of the C_n's.)*

Proof. Suppose Y were such. Then the generic fibre $Y'_{e'}$ would be an infinite co-infinite definable subset of $X'_{e'}$, contradicting Lemma 2.4.

Remark 2.6 *It is possible to prove Corollary 2.5 directly from Campana's analysis, and then deduce strong minimality of $X'_{e'}$ without using Hrushovski-Zilber.*

Explanation. We give a sketch. First let us prove Corollary 2.5, following more or less lines pointed out to us by both Moosa and Campana (whom we thank). Suppose for a contradiction that Y is a 2-dimensional irreducible subvariety of X projecting onto E. On some cofinite subset U of E, the genus g of any irreducible component of the fibre (which is a curve) of Y is constant. Being a subvariety of a torus g could not be 0. If g were strictly greater than 1, then Y would be Moishezon, but then would contain uncountably many curves projecting onto E, contradicting Fact 2.1. Thus g has to equal 1, and so for all but finitely many $e \in E$, each irreducible component of Y_e is a translate of a subtorus of X_e. This easily contradicts Lemma 2.2.

Let us now deduce that $X'_{e'}$ is strongly minimal. We know from Lemma 2.2 that $X'_{e'}$ is a connected e'-definable group with no proper infinite definable subgroups. Also by considering torsion, $acl(e') \cap X'_{e'}$ is infinite. By [14] for example, $acl(e') \cap X'_{e'}$ is an elementary substructure of $X'_{e'}$. So if $X'_{e'}$ were not strongly minimal it would contain an infinite proper irreducible closed subset Z defined over $acl(e')$. This gives rise to a 2-dimensional subvariety of X projecting onto E, contradicting the first paragraph.

Proposition 2.7 *Let p be the generic type (over $\mathcal{A}$) of X. Then $RM(p) = 3$ and $U(p) = 2$.*

Proof. Let a' realize p in X'. Clearly a' is a generic point of $X'_{e'}$ where $e' = q(a')$ is a generic point of E' over $\mathcal{A}$. Let $\phi(x) \in p(x)$. Then by strong minimality of $X'_{e'}$ and E, for all but finitely many $e \in E$, $\{a \in X_e : \phi(a)\}$ is cofinite in X_e. Thus $RM(\phi(x)) = 3$. So $RM(p) = 3$. On the other hand $e' \in dcl(a')$, $U(tp(a'/e')) = 1$ and $U(tp(e')) = 1$. So by the finite U-rank equalities, $U(tp(a'e')) = U(tp(a')) = U(p) = 2$.

Corollary 2.8 *$Th(X)$ has the DOP.*

Proof. First we give an explanation. $Th(X) = T$ is a totally transcendental theory. Shelah [13] defined such a theory to have the DOP (dimensional order property) if there are models M_0, M_1, M_2 of T (elementary substructures of some saturated model of T) with $M_0 \subseteq M_i$ for $i = 1, 2$, such that M_1 is independent from M_2 over M_0, and such that if N is some prime model over $M_1 \cup M_2$ then there is a regular type $p(x) \in S(N)$ which is orthogonal to both M_1 and M_2. T having the DOP implies that T has 2^κ nonisomorphic models of cardinality κ for $\kappa \geq |T| + \omega_1$.

We work in the saturated elementary extension X' of X. Remember that all points of X are named by constants, so a complete type over $\emptyset$ is the same thing as a complete type over X. Actually we work in $(X')^{eq}$ in which E' lives. Let e' be a generic point of E' over $\emptyset$ and let G denote the fibre $X'_{e'}$. Let $r(x)$ be the generic type of G over e'. We know that $r(x)$ is of U-rank 1 (in fact strongly minimal).

Claim. $r(x)$ is orthogonal to $\emptyset$.

Proof. If not, then as in the proof of Theorem 2.5 of [11], there is finite subgroup N of G defined over $acl(e')$, some tuple c from $(X')^{eq}$ which is independent from e' over $\emptyset$, and some c-definable group H such that G/N is definably isomorphic to H. As G/N is strongly minimal, so is H. As $tp(e'/c)$ is finitely satisfiable in X, H is definably isomorphic to a quotient of X'_e by a finite subgroup, for some $e \in E$ (in the base model). But X'_e is not strongly minimal, a contradiction. The claim is proved. (In the proof of the claim we could avoid Theorem 2.5 of [10] and make direct use of the modularity of $X'_{e'}$ as in [8].)

As E is a group, we conclude, as in Corollary 2.6 of [11], that $Th(X)$ has the DOP (even the $ENI - DOP$ insofar as this makes sense).

Finally we show that the example X yields a negative answer to a question raised in [12]. A little more background is necessary before stating the question and result. Recall that $\mathcal{C}$ is the category of (irreducible) compact complex spaces which are bimeromorphic with Kähler manifolds, where a compact complex manifold is said to be Kähler if it carries a Kähler metric, that is a Hermitian metric whose associated 2-form is closed. We can view $\mathcal{C}$ as a many-sorted structure as before. One of the main facts about any member Y of $\mathcal{C}$ is that there is a countable family $\mathcal{R}$ of definable subsets of

$Y, Y \times Y, \ldots$ such that any definable subset of any $Y \times Y .. \times Y$ is definable (with parameters) in the structure $(Y, R)_{R \in \mathcal{R}}$. ω_1 compactness of Y implies saturation of Y as a structure in this full countable language. See [9] for more details and the relation with Douady spaces. Any complex torus is in $\mathcal{C}$ and moreover it follows from [12] that any group definable in $\mathcal{A}$ is definably isomorphic to a group definable in the structure $\mathcal{C}$. The question was whether the same is true for groups definable in the saturated elementary extension $\mathcal{A}'$ of $\mathcal{A}$:
(*) Let G be a connected group definable in $\mathcal{A}'$. Is there a group H definable in $\mathcal{C}'$ such that G is definably (in $\mathcal{A}'$) isomorphic to H?

Any set definable in $\mathcal{A}'$ can be viewed as the generic fibre of a definable family in $\mathcal{A}$. So (*) can be expressed in terms of families of definable groups in $\mathcal{A}$. It is convenient to define a "Kähler family" as follows:

Definition 2.9 *Let Y, Z be irreducible compact complex spaces and $\pi : Y \to Z$ a surjective morphism. We will say that the family (Y, π, Z) (which we think of as the family of fibres) is in $\mathcal{C}$ if there is some $Y_1 \in \mathcal{C}$, some surjective $\pi_1 : Y_1 \to Z_1$, and some compact complex space W surjecting onto both Z and Z_1, such that the spaces $Y \times_Z W$ and $Y_1 \times_{Z_1} W$ are bimeromorphic over W.*

The following is a routine translation:

Lemma 2.10 *Let (Y, π, Z) be a family . Then this family is in $\mathcal{C}$ just if for generic $z' \in Z'$, the fibre $Y'_{z'}$ is bimeromorphic (in $\mathcal{A}'$) to a closed set in $\mathcal{C}'$.*

Let (X, q, E) be the family constructed earlier.

Proposition 2.11 *The family (X, q, E) is not in $\mathcal{C}$, equivalently by Lemma 2.10, the generic fibre $X'_{e'}$ provides a negative answer to (*).*

Proof. The proof goes just like that of Lemma 2.3. We let $X'_{e'}$ be the generic fibre. If $X'_{e'}$ were definably isomorphic to a group H_c say living in $\mathcal{C}'$, then using saturation of $\mathcal{C}$ we find H_{c_0} living in $\mathcal{C}$ such that c and c_0 have the same type over a suitable set of parameters, in a suitable countable sublanguage, and with H_{c_0} definably isomorphic to some $X_e = E_0 \times E_0$. But H_{c_0} has no proper infinite definable subgroups, contradiction.

Questions remain as to whether we can find compact complex spaces $Y \in \mathcal{C}$ with "bad" properties (Morley rank different from U-rank, multidimensionality, DOP).

Finally we point out that there are K3 surfaces which are "totally degenerate" in their canonical language. A K3 surface is a compact complex surface which is simply connected and has trivial canonical class. These are all Kähler. It is pointed out in [9] that a compact Kähler manifold, or even a compact complex space in $\mathcal{C}$ has a canonical countable language, coming from its Douady space. We will say that such a space is totally degenerate if in its canonical language it has no $\emptyset$-definable relations other than Boolean combinations of $x_i = x_j$. In particular it will be ω-categorical (in its canonical language).

Proposition 2.12 *Let X be a compact complex manifold in $\mathcal{C}$ which is strongly minimal, simply connected, and with* $\mathrm{Aut}(X)$ *trivial. Then X is totally degenerate.*

Proof. Note that the assumptions force that $dim(X) > 1$. We first show that there are no strongly minimal subsets of $X \times X$ other than (up to finite) the diagonal and the sets $x = a$, $y = a$. Let Z be a strongly minimal subset of $X \times X$ not of this form, and let (a, b) be its generic point (over $\mathcal{A}$ say). Let Z_1 be the irreducible subvariety of $X \times X$ of which (a, b) is the generic point. Let π_i for $i = 1, 2$ be the two coordinate projections from Z to X. Both π_i are surjective and finite-to-one. As the branch locus of π_i has codimension 1, our assumptions imply that each π_i is an unramified covering. The simply connectedness assumption implies that each π_i is biholomorphic. Thus Z_1 is the graph of an automorphism of X, which by our assumptions must be trivial. So $a = b$, a contradiction.

It follows quite easily that any definable subset of X^n is definable from equality together with names for elements of X. As we are working in the canonical language for X, it follows that X is totally degenerate.

¿From section 5 of [1], we understand that there are K3 surfaces satisfying the hypotheses of the proposition above.

We expect that it is not difficult to show, using similar methods together with the classification of compact complex surfaces, that any 2-dimensional

U-rank 1 type in $\mathcal{C}$ which is trivial is the generic type of a strongly minimal ω-categorical set. We conjecture that the same is true for any trivial U-rank 1 type in $\mathcal{C}$.

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