Differential Arcs

A report on joint work with Rahim Moosa and Anand Pillay

Thomas Scanlon

scanlon@math.berkeley.edu

Banff, 15 March 2004

Differential fields

Definition 1 A *derivation* ∂ on a field K is a function $\partial : K \to K$ satisfying $\partial(x + y) = \partial(x) + \partial(y)$ and $\partial(xy) = x\partial(y) + y\partial(x)$ universally.

The theory of fields of characteristic zero with n commuting derivations, $\mathrm{DF}_{0,n}$, expressed in the language $\mathcal{L}(+, \times, 0, 1, \partial_1, \ldots, \partial_n)$ has a model completion, $\mathrm{DCF}_{0,n}$, the theory of differentially closed fields of characteristic zero with n commuting derivations. [We can relax the commutation condition somewhat to require only that the Lie algebra generated by the distinguished derivations be finite dimensional.]

Differential fields and stability

Let $\mathbb{U} \models \mathrm{DCF}_{0,n}$ be a differentially closed field.

- $\mathrm{RM}(\mathbb{U}) = U(\mathbb{U}) = \omega^n$
- For $K \leq L \leq \mathbb{U}$ algebraically closed differential subfields and $a \in \mathbb{U}^n$, $a \downarrow_K L \iff$ the ideal $I(a/L) := \{P(x) \in L\{x\} \mid P(a) = 0\}$ of differential polynomials over L vanishing at a is generated by $I(a/K)$.
- The canonical base of $\mathrm{tp}(a/K)$ is the field of definition of $I(a/K)$.
- If $\Delta \leq \oplus_{i=1}^n \mathbb{U}\partial_i$ is a subspace with $\dim_{\mathbb{U}} \Delta = d$, then the field $\mathcal{C}^{\Delta}(\mathbb{U}) := \{x \in \mathbb{U} \mid (\forall \delta \in \Delta) \;\; \delta(x) = 0\}$ has Lascar rank ω^{n-d}.

Zilber dichotomy in differential fields

Theorem 1 (Hrushovski-Sokolović) *Let $(\mathbb{U}, +, \times, \partial) \models DCF_{0,1}$ be an ordinary differentially closed field of characteristic zero. If $X \subseteq \mathbb{U}^n$ is a strongly minimal definable set, then either X is locally modular or there is a finite-to-finite correspondence between X and $\mathcal{C}(\mathbb{U}) = \{x \in \mathbb{U} \mid \partial(x) = 0\}$.*

Proof:

- Using some results on polynomial rings, show that, possibly after removing finitely many points, taking the traces of differential varieties on the Cartesian powers of X as closed sets, X is a Zariski geometry.
- Apply the main dichotomy theorem of Zariski geometries and the classification of interpretable fields in $\mathbb{U}$.

Direct proof via higher order derivatives

The interpretation of the field in non-locally modular Zariski geometries is based on a combinatorial notion of "tangency." In $\mathrm{DCF}_{0,n}$ we have natural geometric notions of tangency. The Pillay-Ziegler proof of Theorem 1 is based on these geometric notions of (higher-order) tangency. In addition, the Pillay-Ziegler proof yields much finer information about finite rank types in differentially closed fields.

Algebraic jets

Definition 2 If X is an algebraic variety over the field K, $a \in X(K)$ is a K-rational point, and $m \in \mathbb{N}$ is a natural number, then the m^{th} *jet space to* X *at* a is $J_m(X)_a(K) := \mathrm{Hom}(\mathfrak{m}_{X,a}/\mathfrak{m}_{X,a}^{m+1}, K)$.

Note: The first jet space is none other than the tangent space.

Properties of jet spaces

- There is a map of algebraic varieties $\pi_m : J_m(X) \to X$ so that for any $a \in X(K)$ one may naturally identify $\pi_m^{-1}\{a\}$ with $J_m(X)_a(K)$. So, each $J_m(X)_a$ is an algebraic group definable from a and a field of definition for X.
- The jet space construction gives a contravariant functor.
- If $X, Y \subseteq Z$ are irreducible varieties and $a \in X(K) \cap Y(K)$ is a common point, then $X = Y \iff J_m(X)_a = J_m(Y)_a \leq J_m(Z)_a$ for every m. (**Proof:** WMA these varieties are affine. If $X \neq Y$, then there is some $f \in I(X) \setminus I(Y)$ or vice versa. By Noetherianity, there is some m for which $f \notin \mathfrak{m}_{Y,a}^{m+1}$. There would then be some $\psi \in J_m(Y)_a$ with $\psi(f + \mathfrak{m}_{Z,a}^{m+1}) \neq 0$, but $\varphi(f) = 0$ for every $\varphi \in J_m(X)_a$.)

Algebraic arcs, interpreting nonreduced rings in fields

Let K be an algebraically closed field.

For each natural number m one may regard the affine line over $K[\epsilon]/(\epsilon^{m+1})$ as a variety over K via the correspondence $\langle x_0, \ldots, x_m \rangle \leftrightarrow \sum_{i=0}^m x_i \epsilon^i$.

If $X = V(f_1, \ldots, f_\ell) \subseteq \mathbb{A}^t$ is an affine scheme over $K[\epsilon]/(\epsilon^{m+1})$ one may find a variety $R_m(X)$ over K whose K-points correspond to the $K[\epsilon]/(\epsilon^{m+1})$-points of X. As before, express the each variable x_i as $x_i = \sum_{j=0}^m x_{i,j}\epsilon^j$. With respect to this substitution, one expands $f_s = \sum f_{s,u}\epsilon^u$ where each $f_{s,u}$ is a polynomial in $\{x_{i,j}\}$. Then, $R_m(X) = V(\{f_{s,u}\})$.

Algebraic arcs, definitions

The m^{th} arc bundle of the algebraic variety X, $\mathcal{A}_m(X)$, is just $R_m(X_{k[\epsilon]/(\epsilon^{m+1})})$ where $X_{k[\epsilon]/(\epsilon^{m+1})}$ is X regarded as a scheme over $K[\epsilon]/(\epsilon^{m+1})$ via base change.

More formally, $\mathcal{A}_m(X)$ represents the functor from the category of K-algebras to the category of sets given by $R \mapsto X(R[\epsilon]/(\epsilon^{m+1}))$.

Algebraic arc spaces

If $\ell \geq m$, then the quotient map $R[\epsilon]/(\epsilon^{\ell}) \to R[\epsilon]/(\epsilon^m)$ corresponds to a natural transformation $\pi_{\ell,m} : \mathcal{A}_\ell \to \mathcal{A}_m$.

In the case of $m = 0$, we see that $\mathcal{A}_0(X) = X$. We write π_ℓ for $\pi_{\ell,0}$. For $a \in X(K)$ we define the m^{th} arc space of X at a to be $\mathcal{A}_m(X)_a(K) := \pi_m^{-1}\{a\}$.

Analyzing algebraic arcs in terms of tangents

The arc spaces are not groups in general, but for any ℓ and $\widetilde{a} \in \mathcal{A}_\ell X(K)$, if $a = \pi(\widetilde{a})$ is a smooth point of X, then $\pi^{-1}_{\ell+1,\ell}\{\widetilde{a}\} \cong T_a X$. Moreover, this identification is functorial.

Arcs determine the variety

Proposition 3 *If $X, Y \subseteq Z$ are irreducible varieties over the algebraically closed field K (of characteristic zero) and $a \in X(K) \cap Y(K)$ is a common point, then $X = Y$ if and only if $\mathcal{A}_m(X)_a = \mathcal{A}_m(Y)_a \subseteq \mathcal{A}_m(Z)_a$ for every m.*

Proof:

- If $(\forall m)\mathcal{A}_m X_a = \mathcal{A}_m Y_a$, then

$$\{\widetilde{a} \in X(K[[\epsilon]]) \mid \pi_\infty(\widetilde{a}) = a\} = \{\widetilde{a} \in Y(K[[\epsilon]]) \mid \pi_\infty(\widetilde{a}) = a\}$$

- Without loss of generality, $Z = \mathbb{A}^\ell$ and there is some $f \in I(Y) \setminus I(X)$.

Proof, continued

- Let $L := K(X)$. Extend the map $k[x_1, \ldots, x_\ell]/I_X \to k$ given by $x \mapsto a$ to a place on L with corresponding valuation v. Extend v to $\widetilde{v}$ on L^{alg}. Note that $(L^{\text{alg}}, \widetilde{v})$ satisfies the first-order property *There is an integral point* $b \in X(\mathcal{O}_{L^{alg}, \widetilde{v}})$ *which specializes to* a *and has* $f(b) \neq 0$.
- By Robinson's QE theorem, $(L^{\text{alg}}, \widetilde{v}) \equiv_K (\bigcup_{\ell \geq 1} K((\epsilon^{\frac{1}{\ell}})), \text{ord}_\epsilon)$.
- So, there is some $b \in X(\bigcup_{\ell \geq 1} K[[\epsilon^{\frac{1}{\ell}}]])$ specializing to a and having $f(b) \neq 0$.
- As $K[[\epsilon^{\frac{1}{\ell}}]] \cong_K K[[\epsilon]]$ we may take $\ell = 1$, contradicting our first observation.

Differential prolongation spaces

If $(\mathbb{U}, \partial_1, \ldots, \partial_n)$ is a differential field and ℓ a natural number, we define $\nabla_\ell : \mathbb{U} \to \mathbb{U}^{N(\ell,n)}$ by $\nabla_\ell(x) := \langle \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}(x) \rangle_{|\alpha| \leq \ell}$.

Definition 4 If $X \subseteq \mathbb{U}^m$ is a subset of the m^{th} Cartesian power of $\mathbb{U}$, then the ℓ^{th} prolongation space of X, $\tau_\ell X$, is the Zariski closure of the set $\{\langle \nabla_\ell(a_1), \ldots, \nabla_\ell(a_m) \rangle \mid \langle a_1, \ldots, a_n \rangle \in X\}$.

In the case that X is an algebraic variety defined over the constants and $n = m = 1$, then $\tau_1 X$ is the Zariski tangent bundle of X.

Differential dimension

Definition 5 If $X \subseteq \mathbb{U}^m$ is a differential variety, then its dimension function $\omega_X : \mathbb{N} \to \mathbb{N}$ is defined by $\omega_X(\ell) := \dim \tau_\ell X$.

Theorem 6 (Kolchin) *To each differential variety X there is a polynomial $K_X \in \mathbb{Q}[x]$ for which $K_X(\ell) = \omega_X(\ell)$ for $\ell \gg 0$. The degree of X, called the typical dimension of X, $m(X) := \deg K_X$, and the leading coefficient of X, called the Δ-dimension, $\dim_\Delta(X)$, are definable invariants of the generic type of X.*

If the $m(X) = 0$, we say that X has finite differential dimension.

Algebraic D-varieties

For the time being we specialize to the case of one derivation.

Definition 7 An *algebraic D-variety* (X, s) over a differential field K is an algebraic variety X given together with a section $s : X \to \tau_1 X$ of the first prolongation space.

If (X, s) is an algebraic D-variety over the differentially closed field $\mathbb{U}$, then $(X, s)^\sharp(\mathbb{U}) := \{x \in X(\mathbb{U}) \mid \nabla(x) = s(x)\}$ is a differential variety of finite differential dimension. Moreover, up to a set of lower dimension, every differetial variety of finite dimension has this form.

Differential jet spaces

To an algebraic D-variety (X, s) and point $a \in (X, s)^\sharp(\mathbb{U})$, one may associate a definable subgroup $J_m(X, s)^\sharp_a$ of $J_m X_a(\mathbb{U})$, what we call the m^{th} jet space of $(X, s)^\sharp$ at a.

There are a number of equivalent ways to do this. We indicate two.

$\mathcal{D}$-modules

Let $\mathcal{D} := \mathbb{U}\langle\partial\rangle$ be the ring of linear differential operators over $\mathbb{U}$ generated by ∂.

By definition, if M is a $\mathcal{D}$-module, then the horizontal subspace is the $\mathcal{C}(\mathbb{U})$-vector space $M^\Delta := \{x \in M \mid \partial \cdot m = 0\}$. [If $\dim_{\mathbb{U}} M < \infty$, then the natural map $M^\Delta \otimes_{\mathcal{C}(\mathbb{U})} \mathbb{U} \to M$ is surjective.]

If M is a $\mathcal{D}$-module, then the $\mathbb{U}$-dual $\check{M}$ of M has a natural $\mathcal{D}$-module structure given by $(\partial \cdot \varphi)(x) := \partial(\varphi(x)) - \varphi(\partial \cdot x)$.

Jet spaces via $\mathcal{D}$-modules

The ideal $\mathfrak{m}_{X,a}$ is a $\mathcal{D}$-module via $\partial \cdot f := f^{\partial} + df \cdot s$.

This action gives the space $V := \mathfrak{m}_{X,a}/\mathfrak{m}_{X,a}^{m+1}$ a $\mathcal{D}$-module structure.

Set $J_m(X,s)_a^{\sharp} := (\check{V})^{\Delta}$.

Jet spaces via D-variety structure on $J_m X_a$

There is a natural comparison map $\varphi : J\tau_1 X \to \tau_1 JX$. Take $J_m(X,s)_a^{\sharp} := (J_m(X)_a, \varphi \circ J_m(s))^{\sharp}$.

Key property of $J_m(X, s)^\sharp$

Theorem 8 *If* (Z, s) *is an algebraic* D*-variety,*
$(X, s \upharpoonright X), (Y, s \upharpoonright Y) \subseteq (Z, s)$ *are irreducible sub* D*-varieties, and*
$a \in (X, s)^\sharp(\mathbb{U}) \cap (Y, s)^\sharp(\mathbb{U})$ *is a common point, then*
$X = Y \Leftrightarrow (\forall m) \;\; J_m(X, s)^\sharp_a = J_m(Y, s)^\sharp_a$.

Differential jet spaces in general

One may adapt the definitions of jet spaces of algebraic D-varieties to general differential varieties, but we do not know whether the adapted spaces determine the differential variety.

We do know that their dimensions can be wrong.

Differential arc spaces

Definition 9 If X is a differential variety over the differential field $(K, +, \times, \partial_1, \ldots, \partial_n)$ and m is natural number, then the m^{th} arc bundle of X is the differential variety $\mathcal{A}_m X$ which represents the set valued functor on the category of differential K-algebras given by $R \mapsto X(R[\epsilon]/(\epsilon^{m+1}))$ where $R[\epsilon]/(\epsilon^{m+1})$ is made into a differential ring by defining $\partial_i(\epsilon) = 0$ for all i.

As before, there are maps $\pi_{\ell+m,m} : \mathcal{A}_{\ell+m} \to \mathcal{A}_m$ and $\mathcal{A}_0$ is the identity. For $a \in X(K)$ we define $\mathcal{A}_m X_a := \pi_\ell^{-1}\{a\}$.

Properties of the differential arcs

- The arc spaces $\mathcal{A}_m X_a$ are analyzable in terms of $T_a X$, Kolchin's differential tangent space.
- If $X, Y \subseteq Z$ are are irreducible differential varieties and $a \in X(K) \cap Y(K)$ is a common *smooth* point, then $X = Y$ if and only if $(\forall m)\ \ \mathcal{A}_m(X)_a = \mathcal{A}_m(Y)_a$.
- For $a \in X$ sufficiently general, $mK_X = K_{\mathcal{A}_m X_a}$.

The canonical base over a realization

Theorem 10 *Let $\mathbb{U} \models DCF_{0,n}$ be a differentially closed field, $k \leq \mathbb{U}$ an algebraically closed differential subfield, and a and c tuples from $\mathbb{U}$. We suppose that c is (interdefinable with) the canonical base of* $\mathrm{tp}(a/k, c)$*. Let $X = V(I(a/k))$ be the differential locus of a over k. Then* $\mathrm{tp}(c/k, a)$ *is internal to $\mathcal{A}_m(X)_a$ for some m. If X has finite differential dimension, then* $\mathrm{tp}(c/k, a)$ *is internal to $\mathcal{C}(\mathbb{U})$.*

Proof of Theorem 10

Proof: Let Y be the locus of a over $k\langle c\rangle$. The canonical base, c, of $\mathrm{tp}(a/k, c)$ is interdefinable with the canonical parameter of Y. As Y is determined by its arc spaces at a, there is some m such that c is interdefinable with the canonical parameter of $\mathcal{A}_m(Y)_a$. As $\mathcal{A}_m(Y)_a \subseteq \mathcal{A}_m(X)_a$, by stability, the set $\mathcal{A}_m(Y)_a$ is defined with parameters from $\mathcal{A}_m(X)_a$.

In the finite differential dimension case, we may work with jet spaces instead.

Regular types

Recall that (in a stable theory) a nonalgebraic stationary type $p \in S(A)$ is *regular* if p is orthogonal to every forking extension.

That is, if $A \subseteq B$, $a, b \models p$, $a \downarrow_A B$ and $b \not\downarrow_A B$, then $a \downarrow_B b$.

If $U(p) = \omega^\beta$, then p is regular. In particular, minimal types are regular.

Dichotomy theorem (and conjecture)

Theorem 11 *Let p be a regular type in a differentially closed field $\mathbb{U}$. Either p is locally modular or there is a definable subgroup $G \leq (\mathbb{U}, +)$ of the additive group having a regular generic type $\mathfrak{g}$ which is nonorthogonal to p.*

Conjecture 12 *If p is a non-locally modular regular type in a differentially closed field, then p is nonorthogonal to the generic type of a definable field.*

Key technical lemma: From non-local modularity to nonorthogonality to a large type in a group

Let p be a regular type in a differentially closed field. We suppose that p has minimal differential type in the sense that if q is a regular type and $q \not\perp p$, then the typical dimension of the locus of q is at least that of p.

Lemma 13 *If p is not locally modular, then there is a type q and a definable subgroup $G \leq (\mathbb{U}^\ell, +)$ for which*

- $p \not\perp q$
- $q(x) \vdash x \in G$
- *p, q and G all have the same typical dimension*

Sketch of a proof of Lemma 13

As p is not locally modular, we can find tuples of realizations a and c of p and an algebraically closed $k \leq \mathbb{U}$ for which c is the canonical base of $\mathrm{tp}(a/k, c)$, $r := \mathrm{tp}(c/k, a)$ is regular and nonorthogonal to p, $w_p(a/k) = 2$, and $w_p(a/k, c) = 1$.

Let X be the locus of a over k. Then, r is internal to $\mathcal{A}_m(X)_a$ for some m. So, one finds a type $q' \vdash \mathcal{A}_m(X)_a^\ell$ in which r is internal.

Using the analysis of $\mathcal{A}_m(X)_a$ in terms of $T_a(X)$ and facts about typical dimension, one recovers a type q satisfying the conclusion.

Structure theorem for differential vector groups

By a differential vector group we mean a definable group G which is definably isomorphic to a subgroup of $\mathbb{U}^\ell$ for some ℓ.

Theorem 14 *Every differential vector group G admits a composition series $0 = G_0 < G_1 < \ldots < G_m = G$ for which the successive subquotients G_{i+1}/G_i have regular generic types.*

The key step in the proof is the observation that if $H \leq \mathbb{U}^\ell$, then via the natural identification of $\mathbb{U}^\ell$ with its tangent space at the origin, H is identified with its own tangent space.

Questions

- When $c = \mathrm{Cb}(a/k, c)$ is $\mathrm{tp}(c/k, a)$ internal to the non-locally modular regular types in general?
- Is there a definable subgroup $G \leq (\mathbb{U}, +)$ of the additive group having a regular generic type $\mathfrak{g}$ and a quasiendomorphism of positive $\mathfrak{g}$-weight but no definable field of quasiendomorphisms of positive $\mathfrak{g}$-weight?