DIFFERENTIAL ARCS AND REGULAR TYPES IN DIFFERENTIAL FIELDS

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Abstract. We introduce differential arc spaces in analogy to the algebraic arc spaces and show that a differential variety is determined by its arcs at a point. Using differential arcs, we show that if $(K, +, \times, \delta_1, \ldots, \delta_n)$ is a differentially closed field of characteristic zero with n commuting derivations and $p \in S(K)$ is a regular type over K, then either p is locally modular or there is a definable subgroup $G \leq (K, +)$ of the additive group having a regular generic type that is nonorthogonal to p.

1. Introduction

In many contexts, one may reduce the study of general partial differential equations to the study of *linear* PDEs. For example, when working with germs of meromorphic functions as coefficients and potential solutions, it is possible to construct from a general system of PDEs a corresponding system of linear PDEs whose solvability is equivalent to the solvability of the original system. However, this transformation requires an analytic reparametrization and does not make sense in the category of algebraic differential equations. Nevertheless, there is a technical sense in which the complexity of general algebraic partial differential equations is reducible to that of linear equations.

Recall that a differential field (with n derivations) is a field K given together with n commuting derivations $\partial_i : K \to K$. (One could relax the commutation condition by requiring merely that the Lie algebra spanned by $\partial_1, \ldots, \partial_n$ is finite dimensional. Provided that one can solve enough differential equations on K, by a change of variables one may regard a differential field with initially noncommuting derivations as a differential field with commuting derivations.) By a system of (algebraic) partial differential equations over K we mean a system of equations of the form $F_1(\{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}(x_j)\}) = 0, \ldots, F_\ell(\{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}(x_j)\}) = 0$ where $F_1, \ldots, F_\ell \in K[\{X_{j;\alpha_1,\ldots,\alpha_n}\}_{1 \leq j \leq k; \alpha \in {}^n\mathbb{N}}]$ are polynomials over K in variables appropriate for coordinates $x_1, \ldots, x_\ell$ and their derivatives. A solution to a system of differential equations is given by a differential field extension $L \geq K$ and a point $a \in L^\ell$ for which all of these differential polynomials vanish when evaluated at a.

As with fields and ordinary algebraic equations, one can find differential fields in which every system of differential equations which could have a solution does. Speaking in the language of mathematical logic, in which we shall converse almost exclusively for the statements and proofs of our main theorem, and specializing to characteristic zero, the theory of differential fields of characteristic zero with

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n commuting derivations, $\mathrm{DF}_{0,n}$, has a model completion, $\mathrm{DCF}_{0,n}$, the theory of differentially closed fields of characteristic zero with n commuting derivations. We regard the study of $\mathrm{DCF}_{0,n}$ as being synonomous with the study of algebraic differential equations.

Even for ordinary algebraic differential equations, it is not the case that every such differential equation is analyzable in terms of linear differential equations. For example, there are many differential equations for which if $X \subseteq K$ is the set of solutions to the equation in some differentially closed field K and $L \subseteq K^n$ is a (finite dimensional over the constants) vector space of solutions to a system of linear differential equations, then for any $\Gamma \subseteq X \times L$ defined itself by differential equations, if both projections are surjective, then $\Gamma = X \times L$. Nevertheless, at least in the case of ordinary differential equations, such a situation is always explained by the geometric simplicity of X.

Hrushovski and Sokolović showed in [5] that if a minimal type in $\mathrm{DCF}_{0,1}$ (their proof immediately generalizes to minimal types of finite transcendence degree in $\mathrm{DCF}_{0,n}$) is not locally modular, then it must be nonorthogonal to the generic type of the constant field. Here *locally modular* may be read as *geometrically simple*. This dichotomy theorem lies at the heart of the model-theoretic proof of the characteristic zero function field version of the Mordell-Lang conjecture [4]. When a group is present, local modularity means that all the generic structure comes from definable groups and when a group is absent, local modularity means that all of the structure is essentially binary. Later in this paper we make heavy use of these and related concepts from geometric stability theory. The reader may wish to consult [11] for the technical details.

The Hrushovski-Sokolović proof relies on the central theorem on Zariski geometries of Hrushovski and Zilber [6]. Recently, Pillay and Ziegler found a direct proof of a strengthening of this theorem based on a theory of jet spaces for algebraic D-varieties [12]. It had been hoped that the jet space technology would adapt to higher dimensional systems of partial differential equations, but technical difficulties obstructed a smooth application of these methods. However, arc spaces of differential varieties work beautifully and we employ them to prove a higher dimensional version of the dichotomy theorem. In particular we define the differential arc spaces $\mathcal{A}_m(X)$ of a differential algebraic variety X. The fibre $\mathcal{A}_m(X)_a$ at a point $a \in X$ will not be a definable group, but will be obtained by a sequence of fibrations, each fibre of which is isomorphic to the differential tangent space of X at a, which is a definable group.

The theory $\mathrm{DCF}_{0,n}$ is totally transcendental [9, 15], and in particular, superstable. As such, every type is coordinatized by *regular* types. If $K \models \mathrm{DCF}_{0,n}$ is a differentially closed field, $V \subseteq \oplus_{i=1}^n K\partial_i$ is a d-dimensional subspace of the Lie algebra spanned by the distinguished derivations, and $\mathcal{C}_V := K^V := \{x \in K \mid \partial(x) = 0 \text{ for all } \partial \in V\}$, then the generic type of $\mathcal{C}_V$ has Lascar rank ω^{n-d}. It follows from the Lascar inequalities that this type is regular. We conjecture that every non locally modular regular type in $\mathrm{DCF}_{0,n}$ is nonorthogonal to a generic type of such a constant field. We will prove that *every non-locally modular regular type is nonorthogonal to a regular type which is the generic type of a definable subgroup of the additive group.* As all such groups are defined by linear differential equations, this gives, in particular, a rigorous sense to the assertion that the geometric complexity of general algebraic differential equations is reducible to that of linear differential equations.

Simplying matters somewhat the proof of our main result on regular types in $\mathrm{DCF}_{0,n}$ essentially goes through the following steps:

(I) If X is a differential algebraic variety defined over k, $a \in X$, and c is some tuple such that c is the canonical base of $\mathrm{tp}(a/k, c)$, then there is $m < \omega$ and tuple d from $\mathcal{A}_m(X)_a$ such that $c \in \mathrm{dcl}(k, a, d)$.

(II) If $p = \mathrm{tp}(a/k)$ is a non-locally modular regular type, then there is $b \in p^{eq}$ such that $\mathrm{tp}(b/k)$ has positive p-weight, and such that p is nonorthogonal to $\mathrm{tp}(e/k, b)$ for some e in the differential tangent space $T(X)_b$ of the differential locus X of b at b.

(III) Show that in (II) e can be chosen to be the generic of a regular definable subgroup of $T(X)_b$.

Step (III) is somewhat involved, and depends on some additional data related to Δ-types in the sense of Kolchin, as well as the structure of definable subgroups of powers of the additive group.

2. Differential arcs

In this section we recall the construction of algebraic arc spaces and some of their properties, introduce differential arc spaces, and then demonstrate that differential varieties are determined by their arcs. Arc spaces were introduced by Nash to study resolution of singularities [10]. Kontsevich revived interest in arc spaces by using them as the basis for his theory of motivic integration and Denef and Loeser have systematically used these ideas [2]. The reader may wish to consult [8] for a discussion of the current state of research on arcs and their applications.

We recall the Weil trace construction, following [1]. If $\pi : T \to S$ is map of schemes, then for any scheme Y over T we obtain a set-valued functor on the category of schemes over S via $S' \mapsto Y(S' \times_S T)$. By $Y(S' \times_S T)$ we mean the set of $(S' \times_S T)$-valued points of Y over T; that is, the set of morphisms from $S' \times_S T$ to Y over T. If this functor is representable, then the *Weil restriction of* Y *from* T *to* S, denoted $R_{T/S}(Y)$, is the representing object. That is, $R_{T/S}(Y)$ is a scheme over S such that for any scheme S' over S, the S'-valued points of $R_{T/S}(Y)$ over S can be identified with the $(S' \times_S T)$-valued points of Y over T. The Weil restriction exists under various hypotheses, the relevant one for us is T being finite over S.

We specialise to the case when S is the spectrum of a field k, $T = \mathrm{Spec}(k^{(m)})$ where $k^{(m)} := k[\epsilon]/(\epsilon^{m+1})$ for a natural number m, and $Y = X \otimes_k k^{(m)}$ for X an algebraic variety over k. We view $k^{(m)}$ as a k-algebra under the natural map $a \mapsto a + 0\epsilon + \ldots + 0\epsilon^m$. The m^{th} *arc bundle of* X *over* k is $R_{k^{(m)}/k}(X \otimes_k k^{(m)})$, the Weil restriction of $X \otimes_k k^{(m)}$ from $\mathrm{Spec}(k^{(m)})$ to $\mathrm{Spec}(k)$. We denote it by $\mathcal{A}_m(X/k)$, or just $\mathcal{A}_m X$ when there is no confusion.

Note that $\mathcal{A}_m X$ is an algebraic variety over k, and that for any ring R over k, $\mathcal{A}_m X(R)$ can be naturally identified with $X(R[\epsilon]/(\epsilon^{m+1}))$. Indeed, by definition the R-points of $\mathcal{A}_m X$ correspond to the $(R \otimes_k k^{(m)})$-points of $X \otimes_k k^{(m)}$ over $k^{(m)}$, and the latter are canonically the $R[\epsilon]/(\epsilon^{m+1})$-points of X over k.

In particular, $\mathcal{A}_m X(k)$ i s identified with $X(k^{(m)})$. So in the case that $X \subseteq \mathbb{A}^\ell$ is an affine variety we can write down the equations for $\mathcal{A}_m X \subseteq \mathbb{A}^{\ell(m+1)}$ as follows:

If $X = \mathrm{Spec}(k[x_1, \ldots, x_\ell]/(\{f_j\}_{j \in J}))$, then

$$\mathcal{A}_m X = \mathrm{Spec}(k[\{x_{i,s}\}_{1 \leq i \leq \ell, 0 \leq s \leq m}]/(\{f_{j,t}\}_{j \in J, 0 \leq t \leq m})$$

where $f_{j,t} \in k[\{x_{i,s}\}_{1 \leq i \leq \ell, 0 \leq s \leq m}]$ is defined by the identity

$$f_j((\sum_{t=0}^{m} x_{i,t}\epsilon^t)_{1 \leq i \leq \ell}) = \sum_{t=0}^{m} f_{j,t}\epsilon^t$$

in the ring $k[\{x_{i,s}\}_{1 \leq i \leq \ell, 0 \leq s \leq m}, \epsilon]/(\epsilon^{m+1})$.

If $f : X \to Y$ is a map of algebraic varieties over k, then $\mathcal{A}_m(f) : \mathcal{A}_m X \to \mathcal{A}_m Y$ is the natural map which on k-points is given by f evaluated on $X(k[\epsilon]/(\epsilon^{m+1}))$. More explicitly, working locally assume that $X \subseteq \mathbb{A}^\ell$, $Y \subseteq \mathbb{A}^r$ and $f = (f_1, \ldots, f_r)$. Viewing $b \in \mathcal{A}_m X(k)$ as an element of $\mathbb{A}^\ell(k[\epsilon]/(\epsilon^{m+1}))$ we have that $\mathcal{A}_m(f)(b) = (f_1(b), \ldots, f_r(b))$ where the $f_i(b)$ are computed in the ring $k[\epsilon]/(\epsilon^{m+1})$.

For $\ell > m$, the quotient map $k^{(\ell)} \to k^{(m)}$ corresponds to a natural transformation $\rho_{\ell,m} : \mathcal{A}_\ell \to \mathcal{A}_m$. Identifying $\mathcal{A}_0$ with the identity we write $\rho_{\ell,0}$ as ρ_ℓ. For $a \in X(k)$, the ℓ^{th} *arc space* $\mathcal{A}_\ell X_a$ *of* X *at* a is the fibre of $\rho_{\ell,X} : \mathcal{A}_\ell X \to X$ over a.

We recall some basic properties of algebraic arc spaces.

Lemma 2.1. *Let X be an algebraic variety over a field k and $a \in X(k)$ a smooth point, then for any pair of natural numbers $\ell > m \geq 0$ the restriction of the map $\rho_{\ell,m} : \mathcal{A}_\ell X \to \mathcal{A}_m X$ to $\mathcal{A}_\ell X_a(k)$ is surjective onto $\mathcal{A}_m X_a(k)$.*

Proof. This is essentially Hensel's Lemma. The problem is local, so we may and do assume that X is affine. As a is a smooth point, we may further assume that $X = V(f_1, \ldots, f_r) \subseteq \mathbb{A}^{d+r}$ where $d = \dim_a X$. Working by induction on ℓ, it suffices to show that if $\widetilde{a} \in X(k[\epsilon]/(\epsilon^{m+1}))$ is a lifting of a, then there is a point $\widehat{a} \in X(k[\epsilon]/(\epsilon^{m+2}))$ lifting $\widetilde{a}$. Let $a' \in \mathbb{A}^{d+r}(k[\epsilon]/(\epsilon^{m+2}))$ be any lifting of $\widetilde{a}$. Letting $f := (f_1, \ldots, f_r)$, note that $f(a') \bmod (\epsilon^{m+1}) = f(\widetilde{a}) = 0$ (working in the ring $k^{(m)}$). So there is $b \in k^r$ such that $f(a') = b\epsilon^{m+1}$. To find $\widehat{a}$ one need only solve $df_a(y) = -b$ in k^{d+r} (which is possible since df_a has rank r), and set $\widehat{a} = a' + y\epsilon^{m+1}$. Indeed, $f(a' + y\epsilon^{m+1}) = f(a') + df_a(y)\epsilon^{m+1} = 0$, and so $\widehat{a} \in X(k[\epsilon]/(\epsilon^{m+2}))$. □

The proof of Lemma 2.1 reveals the structure of the relative arc spaces.

Lemma 2.2. *Let $f : X \to Y$ be a map of algebraic varieties over the field k. Let m be a natural number and $a \in \mathcal{A}_m X(k)$. Let $\widetilde{X}$ be the fibre of $\rho_{m+1,m} : \mathcal{A}_{m+1} X \to \mathcal{A}_m X$ over a, and $\widetilde{Y}$ the fibre of $\rho_{m+1,m} : \mathcal{A}_{m+1} Y \to \mathcal{A}_m Y$ over $\mathcal{A}_m(f)(a)$. Let $\overline{a} := \rho_m(a)$.. Then there are biregular maps $\psi_X : \widetilde{X} \to T_{\overline{a}} X$ and $\psi_Y : \widetilde{Y} \to T_{f(\overline{a})} Y$ so that the following diagram is commutative*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\mathcal{A}_{m+1}(f)} & \widetilde{Y} \\ \psi_X \downarrow & & \downarrow \psi_Y \\ T_{\overline{a}} X & \xrightarrow{df_{\overline{a}}} & T_{f(\overline{a})} Y \end{array}$$

Proof. Working locally, we may write $X = V(g_1, \ldots, g_\ell) \subseteq \mathbb{A}^t$ and have $Y \subseteq \mathbb{A}^s$. Fix $a' \in X(k[\epsilon]/(\epsilon^{m+2}))$ a lifting of a. If $b \in \mathbb{A}^t(k[\epsilon]/(\epsilon^{m+2}))$ is another lifting, then we may write $b = a' + y\epsilon^{m+1}$ for some $y \in k^t$. Let $g := (g_1, \ldots, g_\ell)$. In order to have $b \in X(k[\epsilon]/(\epsilon^{m+2}))$ we need

$$0 = g(b) = g(a' + \epsilon^{m+1}y) = g(a') + dg_{\overline{a}}(y)\epsilon^{m+1} = dg_{\overline{a}}(y)\epsilon^{m+1}.$$

That is, $y \in T_{\overline{a}}X$. Conversely, this equation shows that any such point in $T_{\overline{a}}X$ gives rise to an element of $\widetilde{X}$. Likewise, we identify $\widetilde{Y}$ with $T_{f(\overline{a})}Y$. Writing an element of $\widetilde{X}(k)$ as $b = a' + y\epsilon^{m+1}$, we see that

$$\mathcal{A}_{m+1}(f)(b) = f(a' + y\epsilon^{m+1}) = f(a') + df_{\overline{a}}(y)\epsilon^{m+1},$$

which proves the claimed commutivity of the diagram. □

As a consequence of Lemma 2.2, we see that arcs of dominant maps are themselves dominant. More precisely, we have the following lemma.

Lemma 2.3. *Let $f : X \to Y$ be a dominant map of algebraic varieties over the field k. If $a \in X(k)$ is smooth on X, $f(a) \in Y(k)$ is smooth on Y, and the rank of the differential of f at a is equal to $\dim Y$; then for every natural number m, the map $\mathcal{A}_m(f) : \mathcal{A}_m X_a(k) \to \mathcal{A}_n Y_a(k)$ is surjective.*

Proof. We prove this lemma by induction on m with the case of $m = 0$ being trivial. In the case of $m + 1$, let $y \in \mathcal{A}_{m+1}Y_a(k)$. By induction, there is some $\overline{x} \in \mathcal{A}_m X_a(k)$ such that $\mathcal{A}_m(f)(\overline{x}) = \rho_{m+1,m}(y)$. By Lemma 2.1 there is a point $\widetilde{x} \in \mathcal{A}_{m+1}X_a(k)$ with $\rho_{m+1,m}(\widetilde{x}) = \overline{x}$. By Lemma 2.2, we may identify $\rho^{-1}_{m+1,m}\{\overline{x}\}$ with T_aX, $\rho^{-1}_{m+1,m}\{\rho_{m+1,m}(y)\}$ with $T_{f(a)}Y$, and the restriction of $\mathcal{A}_{m+1}(f)$ to this fibre with df_a. As such, the map is surjective between these fibres so that y is in the range of $\mathcal{A}_{m+1}(f)$ as claimed. □

Note that when the characteristic of k is zero, the hypotheses of Lemma 2.3 hold for sufficiently general a whenever f is dominant.

With the next lemma we note that subvarieties are determined by their arc spaces, at least in characteristic zero. We expect that this result has a straightforward characteristic-free algebraic proof and that it may even be well-known, but we could find no such proof in the literature. Our proof will make use of some model theory of algebraically closed valued fields, due essentially to Robinson [14]. Consider the following 3-sorted language L_{val} for valued fields: the field sort with the language of rings, the value group sort with the language of ordered groups, the residue field sort with the language of rings, and also the value map $| - |$ from the field sort to the value group sort, and a 2-ary map Res from the field sort to the residue field sort, which takes (x, y) to the residue of xy^{-1} (and taking value 0 if $|x| > |y|$. Then in the language L_{val}, (i) the complete theory of an algebraically closed nontrivially valued field (K, Γ, k) is determined by the pair $(char(K), char(k))$, and (ii) the theory of any algebraically closed nontrivially valued field has quantifier-elimination. We refer the reader to Theorem 2.1.1 of [3].

Lemma 2.4. *Let k be an algebraically closed field of characteristic zero and $X, Y \subseteq Z$ irreducible algebraic varieties over k. If $a \in X(k) \cap Y(k)$ is a k-point on both X and Y, then $X = Y$ if and only if $\mathcal{A}_m X_a(k) = \mathcal{A}_m Y_a(k)$ for all $m > 0$.*

Proof. Working locally, we may and do assume that $Z = \mathbb{A}^m$ is an affine space. Suppose that $X \neq Y$. Without loss of generality X is not a subvariety of Y, hence there is $f \in I_Y$, $f \notin I_X$. The function field of X, $k(X)$, is the field of fractions of $k[x_1, \ldots, x_m]/I_X$. Extend the natural evaluation map $k[x_1, \ldots, x_m]/I_X \to k$ given by $\vec{x} \mapsto a$ to a k-place on $k(X)$ and let v be the corresponding valuation. Let w be an extension of v to $L := k(X)^{\mathrm{alg}}$, the algebraic closure of $k(X)$. On the other hand, let $K = \bigcup_{\ell=1}^{\infty} k((\epsilon^{\frac{1}{\ell}}))$, where the union is naturally a direct limit.

Let w' be the natural valuation on K (with valuation ring $\bigcup_{\ell=1}^{\infty} k[[\epsilon^{\frac{1}{\ell}}]]$ and residue field k). As k is algebraically closed of characteristic 0, K is algebraically closed. By points (i) and (ii) preceding the statement of this lemma, the valued fields (L, w) and (K, w') are elementarily equivalent over their common residue field k (in the 3-sorted language mentioned above). By construction, we have a point $b \in X(\mathcal{O}_{L,w})$ for which $Res(b, 1) = a$ and $f(b) \neq 0$. Indeed, b is just $\vec{x} \bmod I_X$. Thus there is $c \in X(\mathcal{O}_{K,w'})$ with $Res(c, 1) = a$ and such that $f(c) \neq 0$. For some ℓ, $c \in X(k[[\epsilon^{\frac{1}{\ell}}]]) \setminus Y(k[[\epsilon^{\frac{1}{\ell}}]])$. As $k[[\epsilon]] \cong_k k[[\epsilon^{\frac{1}{\ell}}]]$, we can find $\alpha \in X(k[[\epsilon]]) \setminus Y(k[[\epsilon]])$ which specialises to a. Considering the finite truncations of α, it follows that for some m, there exists $d \in X(k[\epsilon]/(\epsilon^{m+1})) \setminus Y(k[\epsilon]/(\epsilon^{m+1}))$ which specialises to a. That is, $\mathcal{A}_m X_a(k) \neq \mathcal{A}_m Y_a(k)$. This proves the lemma. □

By a Δ-field we will always mean a field of characteristic zero equipped with n commuting derivations $\Delta = \{\partial_1, \ldots, \partial_n\}$.

The arc space construction is very closely related to that of the prolongation spaces in differential algebraic geometry. Suppose k is a Δ-field. Then the ring $k_m := k[\eta_1, \ldots, \eta_n]/(\eta_1, \ldots, \eta_n)^{m+1}$ may be regarded a k-algebra via the map

$$a \longmapsto \sum_{0 \leq \alpha_1 + \cdots + \alpha_n \leq m} \frac{1}{\alpha_1! \cdots \alpha_n!} \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}(a) \eta_1^{\alpha_1} \cdots \eta_n^{\alpha_n}.$$

For X an algebraic variety over k, the *m^{th} prolongation $\tau_m X$ of X* is the Weil restriction of $X \otimes_k k_m$ from $\mathrm{Spec}(k_m)$ to $\mathrm{Spec}(k)$ where k_m is viewed as a k-algebra in the above manner. That is, $\tau_m X = R_{k_m/k}(X \otimes_k k_m)$. So when $n = 1$ and ∂_1 is a trivial derivation, τ_m and $\mathcal{A}_m$ agree. Thinking in terms of the functors they represent, it is also clear that $\mathcal{A}_r \tau_m$ may be identified with $\tau_m \mathcal{A}_r$. At the level of k-points, both $\mathcal{A}_r \tau_m X(k)$ and $\tau_m \mathcal{A}_r X(k)$ are identified with $X(k_m[\epsilon]/(\epsilon^{r+1}))$.

From the reduction (or quotient) maps $k_\ell \to k_m$ (for $\ell \geq m$) we see that the prolongations form a projective system $\pi_{\ell,m} : \tau_\ell \to \tau_m$. Identifying τ_0 with the identity, we write $\pi_{\ell,0}$ as π_ℓ and have a map $\pi_\ell : \tau_\ell X \to X$. The map on k-points given by $x \longmapsto \sum_{0 \leq \alpha_1 + \cdots + \alpha_n \leq m} \frac{1}{\alpha_1! \cdots \alpha_n!} \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}(x) \eta_1^{\alpha_1} \cdots \eta_n^{\alpha_n}$ gives a Δ-regular section to π_m, which we denote by $\nabla_m : X \to \tau_m X$.

It is sometimes convenient to view the higher prolongations as canonically embedded in the iterated prolongations. That is, instead of $\tau_m X$, one might consider $\tau^m X := \overbrace{\tau \circ \cdots \circ \tau}^{m \text{ times}} X$. The corresponding Δ-regular section $\nabla^m : X \to \tau^m X$ is given by $\nabla^m := \overbrace{\nabla \circ \cdots \circ \nabla}^{m \text{ times}}$. We obtain natural embeddings $\tau_m X \hookrightarrow \tau^m X$ from the map

$$k[\eta_1, \ldots, \eta_n]/(\eta_1, \ldots, \eta_n)^{m+1} \to k[\xi_{1,1}, \ldots, \xi_{1,m}; \ldots; \xi_{n,1}, \ldots, \xi_{n,m}]/(\{\xi_{i,j}\}_{1 \leq i \leq n, 1 \leq j \leq m})^2$$

via $\eta_i \mapsto \sum_j \xi_{i,j}$. Under this embedding, the map $\pi_{\ell,m} : \tau_\ell X \to \tau_m X$ (for $\ell \geq m$) extends to the map $\overbrace{\pi \circ \cdots \circ \pi}^{\ell - m \text{ times}} : \tau^\ell X \to \tau^m X$, which we will also denote by $\pi_{\ell,m}$.

Differentially (or Δ-) closed fields (of characteristic zero) may be characterized as algebraically closed Δ-fields K for which given a projective system of dominant maps of irreducible algebraic varieties over K, $\langle \mu_{\ell,m} : X_\ell \to X_m \rangle$, for which X_ℓ is a closed subvariety of $\tau^{\ell-m}X_m$ and $\mu_{\ell,m}$ is the restriction of $\pi_{\ell-m}$ to X_ℓ, there is a point $a \in X_0(K)$ such that $\nabla^\ell(a) \in X_\ell(K)$ for all ℓ.

Now suppose that X is a Δ-variety which is given as a Δ-closed subset of an algebraic variety $\overline{X}$. We can define the m^{th} prolongation $\tau_m X$ of X, to be the algebraic variety whose k-points (where k is a Δ-closed field over which everything is defined) form the Zariski closure of $\nabla_m X(k)$ in $\tau_m\overline{X}(k)$. Note that X is determined, as a Δ-closed subset of $\overline{X}$, by its *prolongation sequence* $\langle \pi_{\ell,m} : \tau_\ell X \to \tau_m X \mid \ell \geq m\rangle$. Indeed $X(k) = \{a \in \tau_0 X(k) : \nabla_\ell(a) \in \tau_\ell X(k), \text{ for all } \ell \geq 0\}$. Conversely, suppose $\langle X_\ell \subseteq \tau_\ell\overline{X} \mid \ell \geq 0\rangle$ is a sequence of irreducible algebraic subvarieties such that:

(a) $\pi_{\ell+1,\ell}$ restricts to a dominant map from $X_{\ell+1}$ to X_ℓ, and
(b) after embedding $\tau_\ell\overline{X}$ in $\tau^\ell\overline{X}$ and $\tau_{\ell+1}\overline{X}$ in $\tau^{\ell+1}\overline{X}$, $X_{\ell+1}$ is a closed subvariety of τX_ℓ,

then there exists a (unique) Δ-subvariety X of $\overline{X}$ such that $\tau_\ell X = X_\ell$.

In the remainder of this section, we work towards a differential analogue of arc spaces for which Lemma 2.4 will hold true. Given a Δ-variety X over k, one could mimic the Weil trace construction in the category of Δ-schemes over k, and define the $m^{\text{th}}(\Delta\text{-})$arc bundle of X, $\mathcal{A}_m X$, to be object which represents the functor $T \mapsto X \otimes_k k^{(m)}(T \otimes_k k^{(m)})$, where $k^{(m)}$ is made into a Δ-ring by taking ϵ to be Δ-constant. We proceed differently however, and leave it to the reader to check the equivalence of these definitions. Our approach is to assume that X is given to us as a Δ-closed subset of an algebraic variety $\overline{X}$, and then to define $\mathcal{A}_m X$ by defining its projective system of prolongations:

Proposition 2.5. *Let k be a Δ-closed field, $X \subseteq \overline{X}$ an irreducible Δ-subvariety of the algebraic variety $\overline{X}$ over k, and m a natural number. Then*

$$\langle \mathcal{A}_m(\pi_{s,t}) : \mathcal{A}_m\tau_s X \to \mathcal{A}_m\tau_t X \mid s \geq t\rangle$$

forms the prolongation sequence of a Δ-subvariety of $\mathcal{A}_m\overline{X}$.

We define *the* m^{th} arc bundle $\mathcal{A}_m X$ of X *to be this Δ-subvariety.*

Moreover, if $a \in X(k)$ is a smooth point *in the sense that $\nabla_s(a)$ is a smooth point on $\tau_s X(k)$ for each s, and $d(\pi_{s,t})_{\nabla_s(a)}$ has full rank for every $s \geq t$, then $\tau_s(\mathcal{A}_m X_a) = \mathcal{A}_m(\tau_s X)_{\nabla_s(a)}$.*

Proof. It is with respect to the natural identification of $\mathcal{A}_m\tau_s\overline{X}$ with $\tau_s\mathcal{A}_m\overline{X}$ that we regard $\mathcal{A}_m\tau_s X$ as an algebraic subvariety of $\tau_s\mathcal{A}_m\overline{X}$.

Viewing the higher prolongations of $\overline{X}$ as embedded in its iterated prolongations, we have that $\tau_{s+1}X \subseteq \tau(\tau_s X)$ for every s. As $\mathcal{A}_m$ preserves inclusions we have $\mathcal{A}_m\tau_{s+1}X \subseteq \mathcal{A}_m \circ \tau(\tau_s X) = \tau\mathcal{A}_m\tau_s X$. Moreover, the maps $\pi_{s,t} : \tau_s X \to \tau_t X$ are dominant, and so by (the remark following) Lemma 2.3 the maps $\mathcal{A}_m(\pi_{s,t}) : \mathcal{A}_m\tau_s X \to \mathcal{A}_m\tau_t X$ are dominant. Hence $\langle \mathcal{A}_m(\pi_{s,t}) : \mathcal{A}_m\tau_s X \to \mathcal{A}_m\tau_t X \mid s \geq t\rangle$ is the prolongation sequence of a Δ-subvariety $Z \subseteq \mathcal{A}_m\overline{X}$. That is,

$$Z(k) = \{a \in \mathcal{A}_m\overline{X}(k) : \nabla_s(a) \in \mathcal{A}_m\tau_s X(k) \text{ for all } s \geq 0\}$$

and $\mathcal{A}_m\tau_s X = \tau_s Z$ for all s (viewed as subvarieties of $\tau_s\mathcal{A}_m\overline{X}$). We define the *mth arc bundle of* X, $\mathcal{A}_m X$, to be Z, and we have proved the main clause of the proposition.

The map $\rho_m : \mathcal{A}_m\overline{X} \to \overline{X}$ restricts to a surjection $\mathcal{A}_m X \to X$ which we will also denote by ρ_m. For $a \in X(k)$, we define the m^{th} *arc space of* X *at* a, $\mathcal{A}_m X_a$, to be the fibre of this map above a. It should be clear that

$$\mathcal{A}_m X_a(k) = \{b \in \mathcal{A}_m \overline{X}_a(k) : \nabla_s(b) \in \mathcal{A}_m(\tau_s X(k))_{\nabla_s(a)} \text{ for all } s \geq 0\}.$$

Hence, for the "moreover" clause of the proposition, it suffices to show that, viewed as a sequence of algebraic subvarieties of $\langle \tau_s(\mathcal{A}_m \overline{X}_a) : s \geq 0 \rangle$,

$$\langle \mathcal{A}_m(\pi_{s,t}) : \mathcal{A}_m(\tau_s X)_{\nabla_s(a)} \to \mathcal{A}_m(\tau_t X)_{\nabla_t(a)} \mid s \geq t \rangle$$

is the prolongation sequence of *some* Δ-subvariety of $\mathcal{A}_m \overline{X}_a$. But the latter is clear once we observe that by Lemma 2.3 and our suffciently general choice of a, $\mathcal{A}_m(\pi_{s,t}) : \mathcal{A}_m(\tau_s X)_{\nabla_s(a)} \to \mathcal{A}_m(\tau_t X)_{\nabla_t(a)}$ is surjective. □

If we wish to emphasize that the arc bundle given by Proposition 2.5 is a Δ-variety, we will write $\mathcal{A}_m^\Delta X$. It should be noted that if X is an *algebraic* variety over the Δ-field k, then $\mathcal{A}_m^\Delta X$ is just the algebraic arc bundle $\mathcal{A}_m X$ considered as a Δ-variety.

From Proposition 2.5 we may derive several useful corollaries.

Lemma 2.6. *Suppose X and Y are irreducible Δ-subvarieties of an algebraic variety over a Δ-closed field k, and $a \in X(k) \cap Y(k)$ is a common smooth point. Then $X = Y$ if and only if $\mathcal{A}_m X_a(k) = \mathcal{A}_m Y_a(k)$ for all $m > 0$.*

Proof. If $\mathcal{A}_m X_a(k) = \mathcal{A}_m Y_a(k)$, then $\tau_s(\mathcal{A}_m X_a)(k) = \tau_s(\mathcal{A}_m Y_a)(k)$, and so by the "moreover" clause of Proposition 2.5 $\mathcal{A}_m(\tau_s X)_{\nabla_s(a)}(k) = \mathcal{A}_m(\tau_s Y)_{\nabla_s(a)}(k)$ for all $s \geq 0$. Thus, for every m and s we have $\mathcal{A}_m(\tau_s X)_{\nabla_s(a)} = \mathcal{A}_m(\tau_s Y)_{\nabla_s(a)}$. By Lemma 2.4, this implies that X and Y have the same prolongation sequences and are therefore equal. □

Lemma 2.7. *Let $X \subseteq \mathbb{A}^\ell$ be a Δ-subvariety of affine space. The first Δ-arc bundle $\mathcal{A}_1 X$ of X is naturally isomorphic to the Δ-tangent bundle TX of X.*

Moreover, suppose $a \in X$ is a smooth point, $\overline{X}$ is the Zariski closure of X in $\mathbb{A}^\ell$, and m is a positive integer. Then the map given by Lemma 2.2 which identifies the fibres of $\mathcal{A}_{m+1}\overline{X}_a \to \mathcal{A}_m \overline{X}_a$ with $T_a\overline{X}$ restricts to an isomorphism of the fibres of $\mathcal{A}_{m+1} X_a \to \mathcal{A}_m X_a$ with $T_a X$.

Proof. The Δ-tangent bundle was introduced by Kolchin in [7]. Note that we can and do systematically identify the first arc bundle of an *algebraic* variety with it's tangent bundle.

Let k be a Δ-closed field over which everything is defined and suppose $a \in X(k)$ and $b \in T_a\overline{X}(k)$. By definition, $(a, b) \in \mathcal{A}_1 X(k)$ if and only if $\nabla_r(a, b) \in T(\tau_r X)(k)$ for all $r \geq 0$, where $T(\tau_r X)$ is viewed as an algebraic subvariety of $\tau_r(T\overline{X})$ under the identification of $T(\tau_r \overline{X})$ with $\tau_r(T\overline{X})$. Note that under this last identification, $\nabla_r(a, b)$ becomes $(\nabla_r a, \nabla_r b)$. That is, $(a, b) \in \mathcal{A}_1 X(k)$ if and only if for all $r \geq 0$

$$\sum_{1 \leq i \leq n^r \ell(r+1)} \frac{\partial P}{\partial x_i}(\nabla_r a) \cdot [\nabla_r b]_i \ = \ 0$$

for all P in the defining ideal of $\tau_r X$, where $[\nabla_r b]_i$ denotes the ith co-ordinate of $\nabla_r b$. These are exactly the equations given by Kolchin in section VIII.2 of [7] for the Δ-tangent bundle.

The "moreover" clause now follows by inspecting the map given in Lemma 2.2. In particular, if $c \in \mathcal{A}_m X_a(k)$ and $r \geq 0$, then by Proposition 2.5 $\nabla_r c \in \mathcal{A}_m(\tau_r X)_{\nabla_r a}$, and the following diagram commutes:

$$\begin{array}{ccc} (\mathcal{A}_{m+1}\overline{X}_a)_c & \xrightarrow{\nabla_r} & (\mathcal{A}_{m+1}\tau_r\overline{X}_{\nabla_r a})_{\nabla_r c} \\ \downarrow & & \downarrow \\ T_a\overline{X} & \xrightarrow{\nabla_r} & T_{\nabla_r a}(\tau_r\overline{X}) \end{array}$$

where the vertical arrows are the maps given by Lemma 2.2 (applied to $\overline{X}$ and $\tau_r\overline{X}$ respectively). It follows that $(\mathcal{A}_{m+1}X_a)_c$ is identified with T_aX, as desired. □

Finally in this section, let us recall Kolchin's notions of Δ-type and typical Δ-dimension. This material is from section 0.3 of [7]. If a is some finite tuple from some Δ-closed field k, and F is a Δ-subfield of k, then there is a numerical polynomial $K_{a/F}(y)$ such that for sufficiently large natural numbers r, $K_{a/F}(r)$ is the transcendence degree of $F(\nabla_r(a))$ over F. The degree of $K_{a/F}$ is called the Δ-*type* of a over F and the leading coefficient of $K_{a/F}$ is called the *typical* Δ-*dimension* of a over F, which we write here as $\dim_\Delta(a/F)$. Note that Δ-type zero corresponds to having finite Δ-transcendence degree. These two quantities are Δ-birational invariants of a over F, namely if b generates over F the same Δ-field as a, then a and b have the same Δ-type and typical Δ-dimension over F.

Working now in a universal Δ-closed field $\mathbb{U}$, if X is an F-irreducible Δ-variety, then define the Δ-type and typical Δ-dimension of X to be those of a over F where a is a generic point of X over F. Likewise we define the Kolchin polynomial K_X of X to be $K_{a/F}$. Note that the transcendence degree of $F(\nabla_r(a))$ over F is the the dimension of the variety $\tau_r(X)$. Hence if $Y \subseteq X$ is a Δ-subvariety with the same Kolchin polynomial then $Y = X$.

Corollary 2.8. *Let $X \subseteq \mathbb{A}^\ell$ be a Δ-subvariety of affine space and m a positive integer. Then $K_{\mathcal{A}_mX} = (m+1)K_X$. Thus, for $a \in X(k)$ smooth, the Δ-type of X and of $\mathcal{A}_mX_a$ are the same.*

Proof. By Proposition 2.5, $\tau_r\mathcal{A}_mX = \mathcal{A}_m\tau_rX$. But for an algebraic variety Y, $\dim\mathcal{A}_mY = (m+1)\dim Y$. □

By Lemma 2.7 one may identify $\mathcal{A}_1X$ with the Δ-tangent bundle TX of X. Specializing Corollary 2.8 to the case of $m = 1$ we see that for $a \in X(k)$ sufficiently general $K_X = K_{T_aX}$. This fact is an old theorem of Kolchin [7].

3. Dichotomy theorem

Here we complete the promised proof that if p is a non-locally modular regular type in a Δ-closed field, then p is nonorthogonal to a regular generic type of a definable subgroup of the additive group.

In what follows, we work inside a fixed universal Δ-closed field $\mathbb{U}$. Both the statement of our main results as well as the methods depend heavily on the machinery of stability theory and its meaning in differential fields. The reader is referred to [13] and [11], but we recall some of the key notions.

We consider $\mathbb{U}$ as a structure in the language of rings together with $\partial_1, \dots, \partial_n$. The first order theory of $\mathbb{U}$ (namely $\mathrm{DCF}_{0,n}$) is ω-stable, so stable, and has quantifier-elimination and elimination of imaginaries. Also $\mathbb{U}$ is a saturated model. Stability provides a notion of independence: $a \downarrow_A b$ (read as $\mathrm{tp}(a/Ab)$ *does not fork over* A) where a, b are tuples and A is a set. In our context the meaning is: the differential fields generated by Aa and Ab are algebraically disjoint over the differential field generated by A. A complete type $p(x) \in S(A)$ is *stationary* if it has a unique nonforking extension over any B containing A. In our context, any type over an algebraically closed differential field is stationary. We say that stationary types p, q (over possibly different sets of parameters) are *orthogonal*, written $p \perp q$ if for any set C of parameters containing $\mathrm{dom}(p)$ and $\mathrm{dom}(q)$, if a and b realize the nonforking extensions of p, q respectively, to C, then $a \downarrow_C b$. The (stationary) type p is said to be *regular* if it is orthogonal to all its forking extensions. If $p(x) \in S(A)$ is a regular type, its set of realizations forms a pregeometry with respect to forking over A. So for a a tuple of realizations of p, $\dim(a)$ makes sense. The regular type $p(x) \in S(A)$ is said to be locally modular if (after possibly replacing $p(x)$ by a nonforking extension $p'(x) \in S(B)$), the corresponding pregeometry is modular, meaning that for finite-dimensional closed sets of realizations of p, X, Y say, $\dim(X) + \dim(Y) - \dim(X \cap Y) = \dim(X \cup Y)$. If $p(x)$ is the generic type of a definable field then p is non locally modular. The optimal result in our context ($\mathrm{DCF}_{0,n}$) would be that any non locally modular regular type is nonorthogonal to the (regular) generic type of a definable field (which would have to be a field of constants). This is the case for types of Δ-type zero.

In addition to the afore-mentioned notions, we make use below of various other notions such as domination equivalence, p-weight, and semiregular types. Chapter 7 of [11] deals with this material.

If $p = \mathrm{tp}(a/A)$, by $m(p)$ we mean the Δ-type of a over the Δ-field generated by the set A (as defined at the end of the last section). Note that $m(p) = m(p')$ for p' a nonforking extension of p. If X is a Δ-variety, $m(X)$ denotes the Δ-type of X. So $m(X) = m(p)$ where p is the generic type of X. For a type p, $\mathrm{loc}(p)$ denotes the Kolchin closure of the set of realization of p. We often use $m(a/A)$ or $\mathrm{loc}(a/A)$ to mean $m(\mathrm{tp}(a/A))$ and $\mathrm{loc}(\mathrm{tp}(a/A))$, respectively.

Lemma 3.1. *The Δ-type and typical Δ-dimension are additive in the sense that*

- $m(ab) = \max\{m(a), m(b/a)\}$,
- *if* $m(a) = m(b/a)$ *then* $\dim_\Delta(ab) = \dim_\Delta(a) + \dim_\Delta(b/a)$, *and*
- *if* $m(a) > m(b/a)$ *then* $\dim_\Delta(ab) = \dim_\Delta(a)$.

Proof. As the dimension functions are additive, the lemma follows by computing the degree and leading coefficient of the sum of two polynomials. □

We begin by investigating some relations between Δ-type and regularity.

Definition 3.2. A regular type p is Δ-*type minimal* if for any other regular type q, $q \not\perp p \Rightarrow m(q) \geq m(p)$.

Question 3.3. Are there regular types in Δ-closed fields which are not Δ-type minimal?

We also consider a related, though distinct, minimality property for Δ-varieties.

Definition 3.4. A Δ-variety X is Δ-*type minimal* if $m(Y) < m(X)$ for any proper Δ-subvariety $Y \subset X$.

Lemma 3.5. *Let r be a type and suppose that $X := \mathrm{loc}(r)$ is Δ-type minimal. Then r is a regular type.*

Proof. Our hypothesis implies already that r is stationary as if r were not stationary, then X would have more than one component of Δ-type $m(X)$.

Let a realize a forking extension (to some algebraically closed Δ-field k) of r and b realize the nonforking extension of r to k. As $\mathrm{loc}(ab/k)$ maps dominantly to X via the projection to the second coordinate, we see that $m(ab/k) \geq m(X)$. However, if $a \not\perp\!\!\!\perp_k b$, then we would have $m(b/k, a) < m(X)$ and $m(a/k) < m(X)$ so that $m(ab/k) < m(X)$. □

In the next lemma, we simply observe that the analysis behind the decomposition (up to domination equivalence) of a type as a product of regular types may be accomplished in such a way that the resulting regular types have Δ-type no more than that of the original type.

Lemma 3.6. *Let q be any stationary type. Then there is a finite sequence of regular types $r_1, \ldots, r_\ell$ such that $m(r_i) \leq m(q)$ for all i and q is domination equivalent to $r_1 \otimes \cdots \otimes r_\ell$.*

Proof. It is enough to show that any regular type r which is nonorthogonal to q is nonorthogonal to some regular r' with $m(r') \leq m(q)$. Suppose r is such. So (after passing to nonforking extensions over some k) there are realizations a of q and b of r such that $a \not\perp\!\!\!\perp_k b$. Let c be the canonical base of $\mathrm{tp}(a/k, b)$. Then $c \notin acl(k)$ and $c \in acl(k, b)$. Thus $r' = tp(c/k)$ is regular and nonorthogonal to r. On the other hand c is contained in the definable closure of k together with a finite sequence of realizations of q. Hence by Lemma 3.1, $m(r') \leq m(q)$. □

Corollary 3.7. *If p is a Δ-type minimal regular type, then for any (not necessarily regular) stationary type r, $r \not\perp p \Rightarrow m(r) \geq m(p)$.*

Proof. Suppose for a contradiction that $m(r) < m(p)$. Using Lemma 3.6 find regular types $r_1, \ldots, r_\ell$ with $m(r_i) \leq m(r) < m(p)$ for $i \leq \ell$ and $r \mathrel{\underline{\sqcup}} r_1 \otimes \cdots \otimes r_\ell$. As $p \not\perp r$, $p \not\perp r_i$ for some i; but this contradicts the Δ-type minimality of p. □

Definition 3.8. A Δ*-vector group* is a group H definable in $\mathbb{U}$ which is definably isomorphic to a subgroup of some Cartesian power of the additive group.

We note that the class of Δ-vector groups is closed under taking definable subgroups and quotients. We also note that because every Δ-vector group is a vector space over the field of Δ-constants, every Δ-vector group is divisible and therefore connected.

Lemma 3.9. *Let G be a Δ-vector group. Then the Δ-tangent space of G at its origin is definably isomorphic to G. Moreover, if $H < G$ is a Δ-subgroup, then the restriction to H of the isomorphism between G and T_0G is an isomorphism between H and T_0H.*

Proof. Present G as a Δ-closed subgroup of ${\mathbb{G}_a}^g$ for some g. For each $r \geq 0$, $\tau_r({\mathbb{G}_a}^g)$ is again some cartesian power of the additive group. Let $\phi_r : \tau_r({\mathbb{G}_a}^g) \to T\tau_r({\mathbb{G}_a}^g)$ be the map given by $x \mapsto (0, x)$ which identifies $\tau_r({\mathbb{G}_a}^g)$ with its tangent space at the origin, $T_0\tau_r({\mathbb{G}_a}^g)$. As an algebraic subgroup of $\tau_r({\mathbb{G}_a}^g)$, the defining ideal of $\tau_r G$ is generated by (homogeneous) linear polynomials, and hence its tangent space at the origin is given by the same polynomials. That is, each ϕ_r restricts to an isomorphism from $\tau_r G$ to $T_0(\tau_r G)$. Recall that under the natural identifications $T_0(\tau_r G) = \mathcal{A}_1(\tau_r G)_0 = \tau_r(\mathcal{A}_1 G_0)$ and $\mathcal{A}_1 G_0 = T_0 G$. So $\langle \phi_r \mid r \geq 0\rangle$ identifies the prolongation sequence of G with the prolongation sequence of T_0G, and hence identifies G with T_0G as desired. The "moreover" clause follows by our construction of the isomorphism. □

Corollary 3.10. *Suppose that G is a Δ-vector group such that $m(H) < m(G)$ for any proper definable subgroup $H < G$. Then $m(X) < m(G)$ for all proper Δ-subvarieties $X \subset G$. In particular, the generic type of G is regular.*

Proof. Let $X \subseteq G$ be a Δ-type minimal Δ-subvariety of G with $m(X) = m(G)$. After translating X we may assume that $0 \in X$ is smooth. By Corollary 2.8 (which in this case is a theorem of Kolchin [7]), $m(T_0X) = m(X) = m(G)$. Visibly, $T_0X \leq T_0G$, and by Lemma 3.9 $T_0G \cong G$. Thus, $T_0X = T_0G$. By Corollary 2.8 again, the Kolchin polynomial of X and of T_0X agree (as do those of G and T_0G). Thus, $X = G$, as desired. By Lemma 3.5 the generic type of G is regular. □

We now analyze the relation between arc spaces and non local modularity. First we point out that the arcs give us information about canonical bases.

Lemma 3.11. *Let k be a small algebraically closed Δ-field and a and c tuples. Let $X := \mathrm{loc}(a/k)$. Suppose that $c = \mathrm{Cb}(a/k, c)$. Then there is an integer $m \geq 0$ and a tuple d from $\mathcal{A}_m X_a$ with $c \in \mathrm{dcl}(k, a, d)$.*

Proof. Let $Y := \mathrm{loc}(a/k, c)$. We know that c is interdefinable with the canonical parameter for Y. By Lemma 2.6, the latter is interdefinable over $\mathrm{dcl}(k, a)$ with the sequence of canonical parameters of $\mathcal{A}_m Y_a$ (considered as subsets of $\mathcal{A}_m X_a$). By stability, each $\mathcal{A}_m Y_a$ is definable with parameters from $\mathcal{A}_m X_a$. □

In general, arc spaces are not groups, but they are analyzable in terms of groups.

Lemma 3.12. *Let X be a Δ-variety over a Δ-closed subfield k and $a \in X(k)$ a smooth point. Let $b \in \mathcal{A}_m X_a(k)$ be a point in an arc space of X over a. Then there are elements $b_1, \ldots, b_m = b$, each in the definable closure of $k \cup \{a, b\}$ such that each b_i is in some $k \cup \{a, b_{i-1}\}$-definable principal homogeneous space for T_aX.*

Proof. By (the proofs of) Lemmas 2.2 and 2.7 each fibre of $\rho_{i+1,i} : \mathcal{A}_{i+1} X_a \to \mathcal{A}_i X_a$ is a principal homogeneous space for T_aX. Set $b_i := \pi_{m,i}(b)$. □

Lemma 3.13. *Let k be a Δ-closed field and $p \in S(k)$ a Δ-type minimal regular type with $m(p) = m$. If p is not locally modular, then there are a Δ-vector group G and a type q having*

- *$m(q) = m(G) = m$,*
- *$q(x) \vdash x \in G$, and*
- *$p \not\perp q$*

Proof. In this argument, we suppress k.

If p is not locally modular, then we can find $a \in p^{\mathrm{eq}}$ and $c \in p^{\mathrm{eq}}$ with $w_p(c/a) = 1$, $w_p(a/c) = 1$, $w_p(a) = 2$, $c = \mathrm{Cb}(a/c)$, and $r := \mathrm{tp}(c/a)$ is p-semiregular. (See Corollary 8.5.2 of [11].) By Lemma 3.11, if $X = \mathrm{loc}(a)$, then there are some integer $m \geq 0$ and a finite tuple $e = (e_1, \ldots, e_\ell)$ of elements of $\mathcal{A}_m X_a$ so that $c \in \mathrm{dcl}(a, e)$. Since $c \notin \mathrm{acl}(a)$, there is a proper subtuple e' of e and some $i \leq \ell$, such that $c \downarrow_a e'$, but $c \not\downarrow_{ae'} e_i$. Thus r is nonorthogonal to $\mathrm{tp}(e_i/ae')$. Let $b_1, \ldots, b_m = e_i$ be given by Lemma 3.12 applied to $e_i \in \mathcal{A}_m X_a$. Then r is nonorthogonal to $\mathrm{tp}(b_j/ae'b_{j-1})$ for some j. By Lemma 3.12 this in turn implies that r is nonorthogonal to a type q with $q(x) \vdash x \in T_a(X)$.

As $w_p(c/a) = 1$ and r is p-semiregular, we actually have that r is regular and nonorthogonal to p. Thus $p \not\perp q$. Put $G = T_a(X)$. By (the comment following) Corollary 2.8, $m(G) = m$. Thus $m(q) \leq m$. As p is Δ-type minimal, $m(q) = m$ by Corollary 3.7. □

Lemma 3.14. *Let p be a Δ-type minimal regular type. Suppose that there are a Δ-vector group G and a type q such that $p \not\perp q$, $q(x) \vdash x \in G$, and $m(q) = m(G) = m(p)$. Then there is a Δ-vector group whose generic type is regular and nonorthogonal to p.*

Proof. We work by induction on $\mathrm{ord}(G) := \langle m(G), \dim_\Delta(G), U(G)\rangle$.

Claim: We may assume that if $H < G$ is a proper definable subgroup of G, then $m(H) < m(G)$.

Proof of Claim: Suppose that $H < G$ and $m(H) = m(G)$. Let $\pi : G \to G/H$ be the natural quotient map. Note that $\mathrm{ord}(G/H) < \mathrm{ord}(G)$ by Lemma 3.1. Also $\mathrm{ord}(H) < \mathrm{ord}(G)$ since $\dim_\Delta(H) \leq \dim_\Delta(G)$ and $U(H) < U(G)$.

Replacing q with a nonforking extension, we may assume that H is definable over $\mathrm{dom}(q)$. Write $q = \mathrm{tp}(a/A)$. Set $\overline{q} := \mathrm{tp}(\pi(a)/A)$ and $q' := \mathrm{tp}(a/A, \pi(a))$. Let $b \in a+H$ be independent from a over $\{A, \pi(a)\}$. Set $q'' := \mathrm{tp}(a-b/A, b)$. Note that q'' is a translation of the nonforking extension of q' to $A \cup \{b\}$. Using transitivity, one sees that either $p \not\perp \overline{q}$ or $p \not\perp q''$. In either case, we conclude by induction. That is, if $p \not\perp \overline{q}$ then by Corollary 3.7 $m(\overline{q}) \geq m(p)$. As $\overline{q}(x) \vdash x \in G/H$ we have $m(\overline{q}) \leq m(G/H) \leq m(G) = m(p)$. Thus, $m(\overline{q}) = m(p)$ so that the hypotheses of this lemma apply with $\overline{q}$ in place of q and G/H in place of G. Likewise, in the case of $p \not\perp q''$ we may replace q with q'' and G with H. ✠

Let r be the generic type of G. By Corollary 3.10 and the above reduction, r is regular. It remains to show that $p \not\perp r$. Taking nonforking extensions we may assume that $p \not\perp^a q$. Again, we suppress the base parameters. Let $a \models p$ and $b \models q$ with $a \not\downarrow b$. As r is the generic type of G, we can find c_1, c_2 realising r such that $b \in \mathrm{dcl}(c_1, c_2)$. Hence $a \not\downarrow c_1, c_2$. Now suppose, for a contradiction, that $p \perp r$. Then $a \downarrow c_1$ and so $a \not\downarrow_{c_1} c_2$. As $c_2 \models r$, our assumption that $p \perp r$ implies that $c_2 \not\downarrow c_1$. Hence $\mathrm{loc}(c_2/c_1)$ is a proper Δ-subvariety of G and so $m(c_2/c_1) < m$ by Corollary 3.10 and the reduction of Claim 1. As $m = m(G) = m(p)$, Corollary 3.7 implies that $p \perp \mathrm{tp}(c_2/c_1)$. But this contradicts $a \not\downarrow_{c_1} c_2$. So $p \not\perp r$, as desired. □

Corollary 3.15. *Let p be a regular non locally modular type. Then there is a Δ-vector group G whose generic type is regular and non-orthogonal to p.*

Proof. Let p' be a regular type nonorthogonal to p, and of minimal Δ-type with this property. By the transitivity of nonorthogonality for regular types, p' is Δ-type minimal (in the sense of Definition 3.2). Lemma 3.13 tells us that p' satisfies the hypotheses of Lemma 3.14, which in turn tells us that there is a Δ-vector group G with a regular generic type $\mathfrak{g}_G$ such that $\mathfrak{g}_G \not\perp p'$. Hence $p \not\perp \mathfrak{g}_G$. □

Lemma 3.16. *Let G be a Δ-vector group with regular generic type $\mathfrak{g}_G$. There is a definable subgroup of the additive group itself whose generic type is regular and nonorthogonal to $\mathfrak{g}_G$.*

Proof. Realize G as a definable subgroup of ${\mathbb{G}_a}^g$ for some g. Now one of the g projections of G to $\mathbb{G}_a$, say π must have infinite image. Let a realize the generic type of G (over the defining set k say of parameters). Then $\pi(a)$ realizes the generic type of $\pi(G)$. But $tp(\pi(a)/k)$ is also regular. Thus $H = \pi(G)$ is a definable subgroup of the additive group with generic type regular and nonorthogonal to the generic type of G. □

Combining all the results of this section, we conclude with our main theorem.

Theorem 3.17. *If p is a regular non locally modular type, then there exists a definable subgroup of the additive group whose generic type is regular and nonorthogonal to p.*

Proof. By Corollary 3.15 there is some Δ-vector group G having a regular generic type $\mathfrak{g}_G$ non-orthogonal to p. By Lemma 3.16 there is a definable subgroup $H \leq \mathbb{G}_a$ of the additive group having a regular generic $\mathfrak{g}_H$ non-orthogonal to $\mathfrak{g}_G$. As nonorthogonality is transitive for regular types, we have $p \not\perp \mathfrak{g}_H$. □

References

[1] S. Bosch, W. Lütkebohmert, and M. Raynaud, **Néron models**, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **21**, Springer-Verlag, Berlin, 1990. x+325 pp.

[2] J. Denef and F. Loeser, Germs of arcs on singular algebraic varieties and motivic integration, *Invent. Math.* **135** (1999), no. 1, 201–232.

[3] D. Haskell, E. Hrushovski, and D. Macpherson, Definable sets in algebraically closed valued fields. Part I: elimination of imaginaries, preprint 2002.

[4] E. Hrushovski, The Mordell-Lang conjecture for function fields, *J. Amer. Math. Soc.* **9** (1996), no. 3, 667–690.

[5] E. Hrushovski and Ž. Sokolović, Minimal types in differentially closed fields, preprint 1992.

[6] E. Hrushovski and B. Zilber, Zariski Geometries, *Journal of the American Mathematical Society*, **9** (1996), 1–56.

[7] E. Kolchin, **Differential algebraic groups**, Pure and Applied Mathematics, **114**, Academic Press, Inc., Orlando, FL, 1985, xvii+271 pp.

[8] E. Looijenga, Motivic measures, Séminaire Bourbaki, Vol. **1999/2000**, Astérisque No. 276, (2002), 267–297.

[9] T. McGrail, The model theory of differential fields with finitely many commuting derivations, *J. Symbolic Logic* **65** (2000), no. 2, 885–913.

[10] J. Nash, Arc structure of singularities, A celebration of John F. Nash, Jr, *Duke Math. J.* **81** (1995), no. 1, 31–38.

[11] A. Pillay, **Geometric Stability Theory**, Oxford University Press, 1996.

[12] A. Pillay and M. Ziegler, Jet spaces of varieties over differential and difference fields, *Selecta Math.* **New ser. 9** (2003), 579–599.

[13] B. Poizat, **A Course in Model Theory: An introduction to contemporary mathematical logic**, Translated from the French by Moses Klein and revised by the author, Universitext. Springer-Verlag, New York, 2000. xxxii+443 pp.

[14] A. Robinson, **Complete Theories**, North-Holland, Amsterdam, 1956.

[15] Y. Yaffe, Model completion of Lie differential fields, *Ann. Pure Appl. Logic* **107** (2001), no. 1-3, 49–86.

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