DENSITY OF ORBITS OF ENDOMORPHISMS OF ABELIAN VARIETIES

DRAGOS GHIOCA AND THOMAS SCANLON

Abstract. Let A be an abelian variety defined over $\bar{\mathbb{Q}}$, and let φ be a dominant endomorphism of A as an algebraic variety. We prove that either there exists a non-constant rational fibration preserved by φ, or there exists a point $x \in A(\bar{\mathbb{Q}})$ whose φ-orbit is Zariski dense in A. This provides a positive answer for abelian varieties of a question raised by Medvedev and the second author in [MS14]. We prove also a stronger statement of this result in which φ is replaced by any commutative finitely generated monoid of dominant endomorphisms of A.

1. Introduction

The following conjecture was raised in [MS14, Conjecture 7.14] (motivated by a conjecture of Zhang [Zha10] for polarizable endomorphisms of projective varieties).

Conjecture 1.1. *Let K_0 be an algebraically closed field of characteristic 0, let X be an irreducible algebraic variety defined over K_0, and let $\varphi : X \longrightarrow X$ be a dominant rational self-map. We suppose there exists no positive dimensional algebraic variety Y and dominant rational map $f : X \longrightarrow Y$ such that $f \circ \varphi = f$. Then there exists $x \in X(K_0)$ whose forward φ-orbit is Zariski dense in X.*

We denote by $\mathcal{O}_\varphi(x)$ the forward φ-orbit, i.e. the set of all $\varphi^n(x)$ for $n \geq 0$, where by φ^n we denote the n-th compositional power of φ. Conjecture 1.1 was proven in [MS14, Theorem 7.16] in the special case $X = \mathbb{A}^m$, and $\varphi := (f_1, \dots, f_m)$ is given by the coordinatewise action of m one-variable polynomials f_i. In this paper we prove Conjecture 1.1 when X is an abelian variety. As a convention, for us, *endomorphisms* of an abelian variety A are self-morphisms of A in the category of algebraic varieties, while the *group endomorphisms* of A are self-morphisms of A in the category of abelian varieties. Our result is the fourth known case of Conjecture 1.1 (besides the case proven by Medvedev and the second author in [MS14], Amerik, Bogomolov and Rovinsky [ABR11] exploited the local dynamical behaviour

D.G. is partially supported by an NSERC grant. T.S. is partially supported by NSF Grant DMS-1363372. This material is based upon work supported by the National Science Foundation under Grant No. 0932078 000 while the authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2014 semester.

of the map φ to prove a special case of Conjecture 1.1 assuming there is a *good* p-adic analytic parametrization for the orbit $\mathcal{O}_\varphi(x)$, and recently, the case when X is a surface was proven in [BGT] using also p-adic methods).

Theorem 1.2. *Let K_0 be an algebraically closed field of characteristic 0. Let A be an abelian variety defined over K_0, and let $\sigma : A \longrightarrow A$ be a dominant map of algebraic varieties. Then the following statements are equivalent:*

1. *there exists $x \in A(K_0)$ such that $\mathcal{O}_\sigma(x)$ is Zariski dense in A.*
2. *there exists no non-constant rational map $f : A \longrightarrow \mathbb{P}^1$ such that $f \circ \sigma = f$.*

The motivation for Conjecture 1.1 comes from two different directions. First, Zhang [Zha10, Conjecture 4.1.6] proposed a variant of Conjecture 1.1 for polarizable endomorphisms φ of projective varietieties X defined over $\bar{\mathbb{Q}}$ (we say that φ is *polarizable* if there exists an ample line bundle $\mathcal{L}$ on X so that $\varphi^*(\mathcal{L}) \overset{\sim}{\to} \mathcal{L}^{\otimes d}$ for some integer $d > 1$). The polarizability condition imposed by Zhang is stronger than the hypothesis from Conjecture 1.1 that φ preserves no non-constant fibration of X. The motivation for the stronger hypothesis appearing in [Zha10, Conjecture 4.1.6] lies in the fact that in his seminal paper [Zha10], Zhang was interested in the arithmetic properties exhibited by the dynamics of endomorphisms of projective varieties. In particular, Zhang was interested in formulating good dynamical analogues of the classical Manin-Mumford and Bogomolov Conjectures, and thus he wanted to use the canonical heights associated to polarizable endomorphisms (previously introduced by Call and Silverman [CS93]). The second motivation for Conjecture 1.1 comes from the fact that its conclusion is known assuming K_0 is an uncountable field of characteristic 0 (see [AC08]). More precisely, in [AC08], Amerik and Campana proved that if φ preserves no non-constant rational fibration, then there exist countably many proper subvarieties Y_i of X so that for each $x \in X(K_0) \setminus \cup_i Y_i(K_0)$, the orbit $\mathcal{O}_\varphi(x)$ is Zariski dense in X. However, if K_0 is countable, then the result of Amerik and Campana leaves open the possibility that each K_0-valued point of X is also a K_0-valued point of some subvariety Y_i for some positive integer i. Hence, Conjecture 1.1 raises a deeper arithmetical question.

We are able to extend Theorem 1.2 to the action of any commutative finitely generated monoid of dominant endomorphisms of an abelian variety. For a monoid S of endomorphisms of an abelian variety A, and for any point $x \in A$, we let $\mathcal{O}_S(x)$ be the S-orbit of x, i.e. the set of all $\psi(x)$, where $\psi \in S$.

Theorem 1.3. *Let K_0 be an algebraically closed field of characteristic 0, and let S be a finitely generated, commutative monoid of dominant endomorphisms of an abelian variety A defined over K_0. Then either there exists $x \in A(K_0)$ such that $\mathcal{O}_S(x)$ is Zariski dense in A or there exists a non-constant rational map $f : A \longrightarrow \mathbb{P}^1$ such that $f \circ \sigma = f$ for each $\sigma \in S$.*

It is reasonable to formulate an extension of Conjecture 1.1 to the setting of a monoid action of rational self-maps on an algebraic variety X. However, there are several additional complications arising from such a generalization even in the case of the dynamics of endomorphisms of an abelian variety A, such as:

(i) Should we impose any restriction on the monoid S? Theorem 1.3 is valid only for finitely generated, commutative monoids, and our method of proof does not seem to extend beyond this case (at least not in the case of arbitrary endomorphisms of an abelian variety A; if S is an arbitrary commuting monoid of dominant group endomorphisms of A, then the conclusion of Theorem 1.3 holds easily). As an aside, note that there are many examples of infinitely generated commutative monoids of endomorphisms of A; simply take infinitely many points of A (linearly independent over $\mathbb{Z}$) and then consider the monoid spanned by translations of A by these points. Once again, the difficulty in extending Theorem 1.3 lies not necessarily with this last example, but with the *mixed* case, i.e., when the endomorphisms in the monoid are compositions of translations with algebraic group endomorphisms of A.

(ii) Assuming there is no non-constant fibration preserved by the entire monoid S, is it true that there exists some $\sigma \in S$ and there exists $x \in A(K_0)$ such that $\mathcal{O}_\sigma(x)$ is Zariski dense in A? We have examples of non-commuting monoids S generated by two group homomorphisms of A such that there is no non-constant fibration preserved by S, even though for *each* $\sigma \in S$ there exists a non-constant fibration preserved by σ. On the other hand, if S is a commutative monoid of group homomorphisms of A, then it is easy to see that the above question has a positive answer.

Finally, we note that Amerik-Campana's result [AC08] was extended in [BGZ] for arbitrary monoids S acting on an algebraic variety X through dominant rational endomorphisms, i.e. if there is no non-constant rational fibration preserved by S, then there exist countably many proper subvarieties $Y_i \subset X$ such that for each $x \in X(K_0) \setminus \cup_i Y_i(K_0)$, the orbit $\mathcal{O}_S(x)$ is Zariski dense in X. Again, similar to [AC08], the result of [BGZ] leaves open the possiblity that if K_0 is countable, then $X(K_0)$ may be covered by $\cup_i Y_i(K_0)$.

Here is the strategy for our proof. By the classical theory of abelian varieties, we know that each endomorphism φ of an abelian variety A is of the form $T_y \circ \tau$ (for a translation map T_y, with $y \in A$) and some (algebraic) group homomorphism τ. Since the endomorphisms φ from the given monoid S commute with each other, we obtain that also the corresponding group homomorphisms τ commute with each other. This gives us a lot of control on the action of the corresponding group homomorphisms τ; in particular, if all endomorphisms from S would also be group homomorphisms, then Theorem 1.3 would follow easily. Essentially, in that special case, the problem would reduce to the following dichotomy: either there exists a positive dimensional algebraic subgroup of A which is fixed by a finite index submonoid of S, or there exists a *single* element σ of S, and there exists an algebraic point x of A whose σ-orbit is Zariski dense in A (essentially, such a point x has the property that the cyclic subgroup generated by x is Zariski dense in A). So, if S consists only of group homomorphisms, the conclusion of Theorem 1.3 holds even in a stronger form. However, if the endomorphisms from S are not all group endomorphisms of A, then the proof is much more complicated. One can still find a necessary and sufficient condition under which there exists a non-constant rational fibration preserved by all elements in S, but that condition is very technical. We note that in our proof we use Faltings' Theorem (originally known as Mordell-Lang Conjecture; see [Fal94]) as follows. For any point $x \in A$, the orbit of x under S is contained in a finitely generated subgroup of A (see Fact 3.6). This yields that the orbit of x under S is not Zariski dense in A if and only if there exist finitely many translates of some proper algebraic subgroups of A which contain the orbit of x. Finally, we note that the exact same proof works to prove a variant of Theorem 1.3 with the abelian variety A replaced by a power of the torus. On the other hand, our proof does not seem to generalize to the case of semiabelian varieties due to the failure of the Poincaré Reducibility Theorem (see Fact 3.2) for semiabelian varieties which are not isogenuous to split semiabelian varieties.

The plan of the paper is as follows. In Section 2 we note several easy statements regarding monoids. We continue by stating some basic facts about abelian varieties in Section 3. Then, in Section 4 and then in Section 5 we prove various reductions of the Theorem 1.3, respectively some auxilliary results needed later. In Section 6 we prove Theorem 1.2 as a way to introduce the reader to the more elaborate argument needed for the proof of Theorem 1.3 (which is completed in Section 7). While Theorem 1.2 is a special case of Theorem 1.3, we have chosen to prove them separately because we believe it is easier for the reader to read first the argument done for a cyclic monoid (Theorem 1.2), which avoids some of the technicalities appearing in the proof of Theorem 1.3.

Acknowledgments. We thank Jason Bell, Thomas Tucker and Zinovy Reichstein for several useful discussions while writing this paper. We also thank the referee for his many useful comments and suggestions.

2. General results regarding monoids

We need some basic facts about finitely generated, commutative monoids. First we need a definition.

Definition 2.1. *Let S be any finitely generated, commutative monoid. For each submonoid $T \subseteq S$, we denote by $\bar{T}$ the set of all $x \in S$ with the property that $xT \cap T \neq \emptyset$.*

So, $\bar{T}$ is the set of all $x \in S$ such that there exist $y, z \in T$ such that $xy = z$. Because T is a submonoid of S, then also $\bar{T}$ is a submonoid of S.

Definition 2.2. *A monoid S is called cancellative if whenever $xy = xz$ for $x, y, z \in S$, then $y = z$.*

We note that a monoid of dominant endomorphisms of a given algebraic variety is a cancellative monoid.

Lemma 2.3. *Let S be a cancellative, commutative monoid generated by the elements $\gamma_1, \dots, \gamma_s$, and let T be a submonoid of S such that $\bar{T} = S$. Then there exists a finitely generated submonoid $T_0 \subset T$ and there exists a positive integer n such that $\gamma_i^n \in \bar{T}_0$ for each $i = 1, \dots, s$.*

Proof. Let $f : \mathbb{N}^s \longrightarrow S$ be the homomorphism of monoids given by $f(e_i) = \gamma_i$, where $e_i \in \mathbb{N}^s$ is the s-tuple consisting only of zeros with the exception of the i-th entry which equals 1. Let U be the set of all $a \in \mathbb{N}^s$ such that $f(a) \in T$, and let H be the subgroup of $\mathbb{Z}^s$ generated by U. Since $\bar{T} = S$, then $H = \mathbb{Z}^s$. Therefore there exist s linearly independent tuples in U; call them $u_1, \dots, u_s$. We claim that the monoid T_0 spanned by $f(u_1), \dots, f(u_s)$ satisfies the conclusion of our Lemma.

Indeed, we first show that $\bar{T}_0 = f(H_0 \cap \mathbb{N}^s)$ where H_0 is the subgroup of $\mathbb{Z}^s$ generated by $u_1, \dots, u_s$. To see this, on one hand, it is clear that $\bar{T}_0 \subseteq f(H_0 \cap \mathbb{N}^s)$. Now, to see the reverse inclusion, note that $\bar{T}_0$ satisfies $\overline{\left(\bar{T}_0\right)} = \bar{T}_0$. Indeed, if $x_1, x_2 \in \bar{T}_0$ and $x \in S$ such that $xx_1 = x_2$, we show that $x \in \bar{T}_0$. We have that there exist $y_i, z_i \in T_0$ such that $x_i y_i = z_i$ for $i = 1, 2$. Then we claim that $x(y_2 z_1) = y_1 z_2$ which would indeed show that $x \in \bar{T}_0$ because $y_2 z_1, y_1 z_2 \in T_0$ (note that T_0 is a submonoid). To see the above equality in the cancellative monoid S, it suffices to prove that $x_1 x y_2 z_1 = x_1 y_1 z_2$. Using that $x_1 x = x_2$, $x_2 y_2 = z_2$ and $x_1 y_1 = z_1$, and that S is commutative, we obtain the desired equality; hence $\overline{\left(\bar{T}_0\right)} = \bar{T}_0$ and thus $\bar{T}_0 = f(H_0 \cap \mathbb{N}^s)$.

Now, since $u_1, \dots, u_s$ are linearly independent over $\mathbb{Z}$ (as elements of $\mathbb{Z}^s$), then H_0 has finite index in $\mathbb{Z}^s$. So, there exists a positive integer n such that $ne_i \in H_0$ for each $i = 1, \dots, s$, and therefore $f(ne_i) = \gamma_i^n \in \bar{T}_0$. ☐

We also need some simple results from linear algebra. The first is a consequence of the Lie-Kolchin triangularization theorem [Kol48].

Fact 2.4. *Let S_0 be a finitely generated, commuting monoid of matrices with entries in $\bar{\mathbb{Q}}$. Then there exists an invertible matrix C (with entries in $\bar{\mathbb{Q}}$) such that for each $A \in S_0$, the matrix $C^{-1}AC$ is upper triangular.*

Fact 2.4 will be used repeatedly throughout our proof. An important consequence of it is that the eigenvalues of each matrix in a commuting monoid S_0 are simply the entries on the diagonal (after a suitable change of coordinates). In particular, this has the following easy corollaries.

Lemma 2.5. *Let S_0 be a commuting monoid of matrices with entries in $\bar{\mathbb{Q}}$, generated by matrices $A_1, \dots, A_s$. Then there exists a positive integer n such that for each matrix A contained in the submonoid of S_0 generated by*

$A_1^n, \dots, A_s^n$, if λ is an eigenvalue of A which is also a root of unity, then $\lambda = 1$.

Proof. The conclusion holds with n being the cardinality of the group of roots of unity contained in the number field L which is generated by all the eigenvalues of the matrices A_i. □

Lemma 2.6. *Let S_0 be a finitely generated, commuting monoid of matrices with the property that for each matrix A in S_0, if λ is an eigenvalue of A which is a root of unity, then $\lambda = 1$. Let $\mathcal{U}_0$ be the set of matrices in S_0 with the property that the eigenspace corresponding to the eigenvalue 1 has the smallest dimension among all the matrices in S_0. Let U_0 be the submonoid generated by $\mathcal{U}_0$. Then $\bar{U_0} = S_0$.*

Proof. Using Fact 2.4, we can choose a basis so that each matrix in S is represented by an upper triangular matrix. Furthermore, we may assume each matrix in $\mathcal{U}$ has the first r entries on the diagonal equal to 1, and none of the other entries on the diagonal are equal to 1 (or to a root of unity). Indeed, we know each matrix in $\mathcal{U}$ has r entries on the diagonal equal to 1; if these entries equal to 1 would not be in the same places of the diagonal for two distinct matrices A and B in $\mathcal{U}$, then for some positive integers m and n we would have that A^mB^n has fewer than r entries equal to 1 on the diagonal. So, indeed the r entries equal to 1 appear in the same position on the diagonal for each matrix in $\mathcal{U}$; so we may assume they are the first r entries, while the remaining $\ell - r$ entries on the diagonal of each matrix in $\mathcal{U}$ is not a root of unity.

Let $A \in \mathcal{U}$. Now, for each matrix $B \in S$, even if there exist entries in the positions $i = r + 1, \dots, \ell$ on the diagonal which are equal to 1, there exists a positive integer n such that the entries on the diagonal of A^nB in the positions $i = r + 1, \dots, \ell$ are not equal to 1. This completes our proof. □

3. Abelian varieties

First we recall several results regarding abelian varieties (see [Mil] or [Mum70] for more details). The setup will be as follows: A is an abelian variety defined over a field K of characteristic 0; since one needs only finitely many parameters in order to define A, then we may assume K is a finitely generated extension of $\mathbb{Q}$. We let $\overline{K}$ be a fixed algebraic closure of K. At the expense of replacing K by a finite extension we may assume that all algebraic group endomorphisms of A are defined over K; we denote by $\mathrm{End}(A)$ the ring of all these endomorphisms. Since the torsion subgroup C_{tor} of any algebraic subgroup $C \subseteq A$ is Zariski dense in C, we conclude that any algebraic subgroup of A is defined over $K(A_{\mathrm{tor}})$. Frequently we will use the following facts.

First, as a matter of notation, always the connected component of an algebraic subgroup B of A is denoted by B^0 (we recall that B^0 is the connected algebraic subgroup of B of maximal dimension).

Fact 3.1. *Let B and C be algebraic subgroups of the abelian variety A. Then $(B+C)^0 = \big(B^0 + C^0\big)$.*

Proof. The algebraic group $B^0 + C^0$ is the image of the connected group $B^0 \times C^0$ under the sum map and is therefore connected. As $B^0 \times C^0$ has finite index in $B \times C$, its image under the sum map has finite index in $B+C$. Hence, $B^0 + C^0 = (B+C)^0$. □

The following result is proven in [Mil, Proposition 10.1].

Fact 3.2 (Poincaré's Reducibility Theorem)**.** *If $B \subseteq A$ is an abelian subvariety of A, then there exists an abelian subvariety $C \subseteq A$ such that $A = B+C$ and $B \cap C$ is finite; in particular A/B and C are isogenous.*

Poincaré's Reducibility Theorem yields that any abelian variety is isogenous with a direct product of finitely many simple abelian varieties, i.e. $A \stackrel{\sim}{\to} A_0 := \prod_{i=1}^r C_i^{k_i}$, where each C_i is simple. Then $\mathrm{End}(A) \stackrel{\sim}{\to} \mathrm{End}(A_0)$ (see also [Mil, Section 1.10]), and moreover $\mathrm{End}(A_0) \stackrel{\sim}{\to} \prod_{i=1}^r M_{k_i}(R_i)$, where $M_{k_i}(R_i)$ is the ring of all k_i-by-k_i matrices with entries in the ring $R_i := \mathrm{End}(C_i)$. For any simple abelian variety C, the ring $R := \mathrm{End}(C)$ is a finite integral extension of $\mathbb{Z}$. Therefore we have the following fact.

Fact 3.3. *Let A be an abelian variety defined over a field of characteristic 0. For each algebraic group endomorphism $\phi : A \longrightarrow A$ there exists a minimal monic polynomial $f \in \mathbb{Z}[t]$ of degree at most $2\dim(A)$ such that $f(\phi) = 0$.*

The following result is proven in [Mil, Corollary 1.2].

Fact 3.4 (Rigidity Theorem)**.** *Each endomorphism $\psi : A \longrightarrow A$ is of the form $T_y \circ \phi$ for some $y \in A$, where $T_y : A \to A$ is the translation map $x \mapsto x + y$, and $\phi \in \mathrm{End}(A)$ is an* algebraic group *endomorphism. In particular, if ψ is dominant, then $\phi : A \longrightarrow A$ is an isogeny. Furthermore, the pair (T_y, ϕ) is uniquely determined by ψ.*

As a simple consequence of Fact 3.4, we obtain.

Lemma 3.5. *Let $\psi_1, \psi_2 : A \longrightarrow A$ be endomorphisms of the form $\psi_i := T_{y_i} \circ \varphi_i$ (for $i = 1, 2$) where $\varphi_i : A \longrightarrow A$ are group endomorphisms. If $\psi_1 \circ \psi_2 = \psi_2 \circ \psi_1$, then $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$.*

The following result is an immediate application of the structure theorem for the ring of group endomorphisms of an abelian variety.

Fact 3.6. *Let S be a finitely generated commutative monoid of endomorphisms of an abelian variety A as an algebraic variety. Then for each point $x \in A$, there exists a finitely generated subgroup $\Gamma \subset A$ containing $\mathcal{O}_S(x)$.*

Proof. Let $\{\gamma_1, \dots, \gamma_s\}$ be a set of generators for S. For each $i = 1, \dots, s$, we let $\gamma_i := T_{y_i} \circ \tau_i$ for some translations T_{y_i} (where $y_i \in A$) and some group endomorphisms τ_i. Let $d := \dim(A)$. Then, by Fact 3.3, for each $i = 1, \dots, s$, there exist integers $c_{i,j}$ such that

$$\tau_i^{2d} + c_{i,2d-1}\tau_i^{2d-1} + \cdots + c_{i,1}\tau_i + c_{i,0} \cdot \mathrm{id} = 0,$$

where id always represents the identity map. Then $\mathcal{O}_S(x)$ is contained in the subgroup $\Gamma \subset A$ generated by $\gamma(x), \gamma(y_1), \dots, \gamma(y_s)$, where γ varies among the finitely many elements of S of the form $\gamma := \gamma_1^{m_1} \circ \cdots \gamma_s^{m_s}$, with $0 \le m_i < 2d$, for each $i = 1, \dots, s$. ☐

The next result is a relatively simple application of Fact 3.2.

Lemma 3.7. *Let $B \subseteq A$ be an algebraic subgroup of the abelian variety A. Then $B \neq A$ if and only there exists a nonzero algebraic group endomorphism $\psi : A \longrightarrow A$ such that $\psi(B) = \{0\}$.*

Proof. Clearly, if $B = A$, then there exists no nonzero endomorphism ψ of A such that $\psi(B) = \{0\}$. Now, assume $B \neq A$. We note that it suffices to prove the existence of $\psi \in \mathrm{End}(A)$ such that $B^0 \subseteq \ker(\psi)$, where B^0 is the connected component of B containing 0 for if $B^0 \subseteq \ker(\psi)$ and $N := [B : B^0]$ is the index of B^0 in B, then $B \subseteq \ker(\phi)$ where $\phi = [N] \cdot \psi$. So, from now on assume B is an abelian subvariety of A. We let $\pi : A \longrightarrow A/B$ be the canonical quotient map. By Fact 3.2, we obtain that there exists an abelian subvariety $C \subseteq A$ and an isogeny $\tau : A/B \longrightarrow C$. So, letting $\iota : C \longrightarrow A$ be the canonical injection map, we get that $\psi := \iota \circ \tau \circ \pi : A \longrightarrow A$ is an endomorphism with the property that $\psi(B) = \{0\}$. We claim that $\psi \neq 0$. Indeed, by construction, the image of ψ is C which is a positive dimensional variety (since $B \neq A$). ☐

The following result is the famous consequence of Mordell-Lang Conjecture proven by Faltings [Fal94].

Fact 3.8 (Faltings' theorem; Mordell-Lang Conjecture)**.** *Let $V \subset A$ be an irreducible subvariety with the property that there exists a finitely generated subgroup $\Gamma \subseteq A(\overline{K})$ such that $V(\overline{K}) \cap \Gamma$ is Zariski dense in V. Then V is a coset of an abelian subvariety of A.*

We will also employ the following easy result.

Lemma 3.9. *Let A be an abelian variety. If $x \in A$ is a point generating a cyclic group which is Zariski dense in A, then for each positive integer ℓ, the cyclic group generated by ℓx is Zariski dense in A.*

Proof. Let H be the Zariski closure of the cyclic group generated by ℓx; then H is an algebraic subgroup of A. Furthermore, because the cyclic group generated by x is Zariski dense in A, then

$$A = \bigcup_{i=0}^{\ell-1} (ix + H) .$$

Since A is connected, we conclude that $H = A$, as desired. ☐

Finally, for any simple abelian variety A defined over a field K of characteristic 0, the action of $\mathrm{Gal}(\overline{K}/K)$ on A_{tor} yields the following result.

Fact 3.10. *The group $\mathrm{Gal}(K(A_{\mathrm{tor}})/K)$ embeds into $\mathrm{GL}_{2d}(\hat{\mathbb{Z}})$, where $d = \dim(A)$ and $\hat{\mathbb{Z}}$ is the ring of finite adéles.*

4. Reductions

Next we proceed with several preliminary results used later in the proof of Theorem 1.2. The following result was proven in the case of a cyclic group S of automorphisms in [BRS10]; we thank Jason Bell for pointing out how to extend the result from [BRS10] to our setting.

Lemma 4.1. *It suffices to prove Theorem 1.3 for a submonoid of S spanned by iterates of each of the generators of S.*

Proof. We consider a finite generating set $\mathcal{U} := \{\gamma_1, \dots, \gamma_s\}$ for the monoid S. We assume S does not fix a non-constant fibration of A (otherwise Theorem 1.3 holds). We let S' be the submonoid of S spanned by the endomorphisms in $\mathcal{U}' := \{\gamma_1^{m_1}, \dots, \gamma_s^{m_s}\}$ (for some positive integers m_i). We assume Theorem 1.3 holds for S'. If also S' does not fix a non-constant fibration, then there exists $x \in A(K_0)$ such that the S'-orbit of x is Zariski dense in A; hence also $\mathcal{O}_S(x)$ is Zariski dense in A. So, it remains to prove that S' cannot fix a non-constant fibration if S does not fix a non-constant fibration.

We assume $f \circ \gamma_i^{m_i} = f$ for some non-constant map $f : A \longrightarrow \mathbb{P}^1$ (for each i). Let S_{rep} be a finite set of representatives for the cosets of S' in S (note that S/S' is a finite group since it is a finite monoid in which each element is invertible); without loss of generality we assume the identity is part of S_{rep}. Let $m := |S_{\mathrm{rep}}|$, and let $S_{\mathrm{rep}} := \{\sigma_1, \dots, \sigma_m\}$. Let $s_1, \dots, s_m$ be the elementary symmetric functions $s_i : (\mathbb{P}^1)^m \to \mathbb{P}^1$ and let $g_i := s_i(f \circ \sigma_1, \dots, f \circ \sigma_m)$ (for $i = 1, \dots, m$) Clearly, γ_i preserves each fibration g_j; hence if one g_j is non-constant, then we are done. If each g_j is a constant, then we obtain a contradiction because $f = f \circ \mathrm{id}$ would be a root of the polynomial (with constant coefficients)

$$X^m - g_1 X^{m-1} + g_2 X^{m-2} + \cdots + (-1)^m g_m = 0.$$

This completes the proof of Lemma 4.1. □

Lemma 4.2. *With the notation as in Theorem 1.3, let T be a submonoid of S such that $\bar{T} = S$. If the conclusion of Theorem 1.3 holds for T, then it holds for S.*

Proof. We assume that there exists no non-constant fibration preserved by all elements of S, and it suffices to prove that there is also no non-constant fibration preserved by the elements of T. Assume, by contradiction that there exists $f : A \longrightarrow \mathbb{P}^1$ such that $f \circ \psi = f$ for each $\psi \in T$. Now, let $\sigma \in S$; then there exist $\psi_1, \psi_2 \in T$ such that $\gamma\psi_1 = \psi_2$. So, using also that S is commutative, we get

$$f \circ \gamma = f \circ \psi_1 \circ \gamma = f \circ \psi_2 = f.$$

Hence f must be constant, as desired. □

Combining Lemmas 4.1 and 4.2 we obtain the following reduction of Theorem 1.3.

Lemma 4.3. *With the notation from Theorem 1.3, assume the monoid S is generated by the maps $\gamma_1, \dots, \gamma_s$. Then it suffices to prove the conclusion of Theorem 1.3 for a finitely generated submonoid T of S with the property that $\gamma_i^n \in \bar{T}$ for each $i = 1, \dots, s$, for some positive integer n.*

5. Auxiliary results

In this Section we present several technical results useful for our proof of Theorems 1.2 and 1.3.

Lemma 5.1. *Let K_0 be an algebraically closed field of characteristic 0. Let $\psi_1, \dots, \psi_s : B \longrightarrow C$ be algebraic group morphisms of abelian varieties, and let $y_1, \dots, y_s \in C(K_0)$. Then there exists $x \in B(K_0)$ such that for each $i = 1, \dots, s$, the Zariski closure of the subgroup generated by $\psi_i(x) + y_i$ is the algebraic group generated by $\psi_i(B)$ and y_i.*

Proof. Let K be a finitely generated subfield of K_0 such that B, C, $\psi_1, \dots, \psi_s$ are defined over K, and moreover, each $y_i \in C(K)$. Without loss of generality, we may assume K_0 is a fixed algebraic closure $\overline{K}$ of K (a priori, K_0 may be a proper extension of $\overline{K}$, and thus showing the conclusion with $\overline{K}$ in place of K_0 suffices).

We let $B = A_1 + \cdots + A_m$ written as a sum of simple abelian varieties.

Let $i = 1, \dots, s$; then $\psi_i(B)$ equals the sum $\psi_i(A_1) + \cdots + \psi_i(A_m)$ (each algebraic group being either simple or trivial). We find an algebraic point $x_i \in \psi_i(B)$ such that the Zariski closure of the cyclic group generated by $x_i + y_i$ is the algebraic group generated by $\psi_i(B)$ and y_i; moreover we ensure that

$$\bigcap_{i=1}^{s} \psi_i^{-1}(\{x_i\})$$

is nonempty (in B). We find x_i as a sum $x_{i,1} + \cdots + x_{i,m}$, where each $x_{i,j} \in \psi_i(A_j)$. If for some j we have $\psi_i(A_j) = \{0\}$, we simply pick $x_{i,j} = 0$. Then our goal is to construct the sequence $\{x_{i,j}\}$ such that for each $j = 1, \dots, m$, the set

$$\bigcap_{i=1}^{s} (\psi_i)|_{A_j}^{-1} (\{x_{i,j}\}) \tag{5.1.1}$$

is nonempty (in A_j). Obviously when $\psi_i(A_j) = \{0\}$, we might as well disregard the set

$$(\psi_i)|_{A_j}^{-1}(\{x_{i,j}\}) = (\psi_i)|_{A_j}^{-1}(\{0\}) = A_j$$

from the above intersection. Let now $j = 1, \dots, m$ such that $\psi_i(A_j)$ is nontrivial. We will show that there exists $x_{i,j} \in \psi_i(A_j)$ such that for any positive integer n we have

$$n x_{i,j} \notin (\psi_i(A_j)) \left(K \left(C_{\mathrm{tor}}, x_{i,1}, \dots, x_{i,j-1} \right) \right). \tag{5.1.2}$$

Claim 5.2. If the above condition (5.1.2) holds for each $j = 1, \dots, m$ such that $\psi_i(A_j) \neq \{0\}$, then the Zariski closure of the cyclic group generated by $x_i + y_i$ is the algebraic subgroup B_i generated by $\psi_i(B)$ and y_i.

Proof of Claim 5.2. Indeed, assume there exists some algebraic subgroup $D \subseteq C$ (not necessarily connected) such that $x_i + y_i \in D(\overline{K})$. Let $j \le m$ be the largest integer such that $x_{i,j} \neq 0$; then we have

$$x_{i,j} \in ((-y_i - x_{i,1} - \cdots - x_{i,j-1}) + D) \cap \psi_i(A_j).$$

Assume first that $\psi_i(A_j) \cap D$ is a proper algebraic subgroup of $\psi_i(A_j)$. Since $\psi_i(A_j)$ is a simple abelian variety, then $D \cap \psi_i(A_j)$ is a 0-dimensional algebraic subgroup of C; hence there exists a nonzero integer n such that $n \cdot (D \cap \psi_i(A_j)) = \{0\}$. Then $nx_{i,j}$ is the only (geometric) point of the subvariety $n \cdot (((-y_i - x_{i,1} - \cdots - x_{i,j-1}) + D) \cap \psi_i(A_j))$ which is thus rational over $K(C_{\mathrm{tor}}, x_{i,1}, \dots, x_{i,j-1})$. But by our construction,

$$nx_{i,j} \notin \psi_i(A_j)(K(C_{\mathrm{tor}}, x_{i,1}, \dots, x_{i,j-1}))$$

which is a contradiction. Therefore $\psi_i(A_j) \subseteq D$ if j is the largest index $\le m$ such that $x_{i,j} \neq 0$ (or equivalently, such that $\psi_i(A_j) \neq 0$). So, $x_i + y_i \in D$ yields now $x_i' + y_i \in D$, where $x_i' := x_{i,1} + \cdots + x_{i,j-1}$. Repeating the exact same argument as above for the next positive integer $j_1 < j$ for which $\psi_i(A_{j_1}) \neq \{0\}$, and then arguing inductively we obtain that each $\psi_i(A_j)$ is contained in D, and therefore $\psi_i(B) \subseteq D$. But then $x_i \in \psi_i(B) \subseteq D$ and so, $y_i \in D$ as well, which yields that the Zariski closure of the group generated by $x_i + y_i$ is the algebraic subgroup B_i of C generated by $\psi_i(B)$ and y_i. $\square$

We just have to show that we can choose $x_{i,j}$ both satisfying (5.1.2) and also such that the above intersection (5.1.1) is nonempty. So, the problem reduces to the following: L is a finitely generated field of characteristic 0, $\varphi_1, \dots, \varphi_\ell$ are algebraic group homomorphisms (of finite kernel) between a simple abelian variety A and another abelian variety C all defined over L, and we want to find $z \in A(\overline{K})$ such that for each positive integer n, and for each $i = 1, \dots, \ell$, we have

$$n\varphi_i(z) \notin \varphi_i(A)\left(L\left(C_{\mathrm{tor}}\right)\right). \tag{5.2.1}$$

Indeed, with the above notation, $A := A_j$, L is the extension of K generated by $x_{i,k}$ (for $i = 1, \dots, s$, and $k = 1, \dots, j-1$), and the φ_i's are the homomorphisms ψ_i's (restricted on $A = A_j$) for which $\psi_i(A_j)$ is nontrivial.

Let d be the maximum of the degree of the isogenies $\varphi_i' : A \longrightarrow \varphi_i(A) \subset C$. In particular, this means that for each $w \in C(\overline{K})$, and for each $z \in A(\overline{K})$ such that $\phi_i(z) = w$ we have

$$[L(z) : L] \le d \cdot [L(w) : L]. \tag{5.2.2}$$

For any subfield $M \subseteq \overline{K}$, we let $M^{(d)}$ be the compositum of all extensions of M of degree at most equal to d.

Claim 5.3. Let L be a finitely generated field of characteristic 0, let C be an abelian variety defined over L, let $L_{\mathrm{tor}} := L(C_{\mathrm{tor}})$, and let d be a positive integer. Then there exists a normal extension of $L_{\mathrm{tor}}^{(d)}$ whose Galois group is not abelian.

Proof of Claim 5.3. As proven in [Tho13], the field L_{tor} is Hilbertian (note that L itself is Hilbertian since it is a finitely generated field of characteristic 0). For each positive integer n, according to [FJ08, Corollary 16.2.7 (a)], there exists a Galois extension L_n of L_{tor} such that $\mathrm{Gal}(L_n/L_{\mathrm{tor}}) \xrightarrow{\sim} S_n$ (the symmetric group on n letters). Assume there exists a abelian extension L_0 of $L_{\mathrm{tor}}^{(d)}$ containing L_n. If $n > \max\{5, d!\}$, we will derive a contradiction from our assumption.

We let $G_1 := \mathrm{Gal}\left(L_0/L_{\mathrm{tor}}^{(d)}\right)$ and $G_0 := \mathrm{Gal}(L_0/L_{\mathrm{tor}})$. Then there exists a surjective group homomorphism $f : G_0 \longrightarrow S_n$. Because G_1 is a normal subgroup of G_0 (and f is a surjective group homomorphism), we get that $f(G_1)$ is a normal subgroup of S_n, and moreover, it is abelian since G_1 is abelian. Because $n \geq 5$, the only proper normal subgroup of S_n is A_n, which is not abelian. Hence, $G_1 \subseteq \ker(f)$, and therefore, f induces a surjective group homomorphism (also denoted by f) from G_0/G_1 to S_n; more precisely, we have a surjective group homomorphism $f : G^{(d)} \longrightarrow S_n$, where $G^{(d)} := \mathrm{Gal}\left(L_{\mathrm{tor}}^{(d)}/L_{\mathrm{tor}}\right)$. But $G^{(d)}$ is a group of exponent $d!$, and so, $S_n = f(G^{(d)})$ is also a group of exponent $d!$, which is a contradiction with the fact that $n > d!$. ☐

Claim 5.3 yields that there exists a point $z \in A(\overline{K})$ which is not defined over a abelian extension of $L\left(C_{\mathrm{tor}}\right)^{(d)}$; i.e., $nz \notin A\left(L\left(C_{\mathrm{tor}}\right)^{(d)}\right)$ for all positive integers n. Hence, $n\phi_i(z) \notin \phi_i(A)\left(L\left(C_{\mathrm{tor}}\right)\right)$ (see (5.2.2)), which concludes the proof of Lemma 5.1. ☐

The next result will be used (only) in the proof of Theorem 1.2.

Lemma 5.4. *It suffices to prove Theorem 1.2 for a conjugate $\gamma^{-1} \circ \sigma \circ \gamma$ of the automorphism σ under some automorphism γ.*

Proof. Since $\mathcal{O}_{\gamma^{-1}\sigma\gamma}(\gamma^{-1}(x)) = \gamma^{-1}\left(\mathcal{O}_\sigma(x)\right)$, we obtain that there exists a Zariski dense orbit of a point under the action of σ if and only if there exists a Zariski dense orbit of a point under the action of $\gamma^{-1} \circ \sigma \circ \gamma$. Also, σ preserves a non-constant fibration $f : A \longrightarrow \mathbb{P}^1$ if and only if $\gamma^{-1}\sigma\gamma$ preserves the non-constant fibration $f \circ \gamma$. ☐

The conclusion of the next result shares the same philosophy as the conclusion of Lemma 5.1: one can find an algebraic point in an abelian variety so that it is *sufficiently* generic with respect to any given set of finitely many points.

Lemma 5.5. *Let K_0 be an algebraically closed field of characteristic* 0*, let $\Gamma \subseteq A(K_0)$ be a subgroup such that* $\mathrm{End}(A) \otimes_{\mathbb{Z}} \Gamma$ *is a finitely generated*

$\mathrm{End}(A)$-*module, and let* $B \subseteq A$ *be a nontrivial abelian subvariety. Then there exists* $x \in B(K_0)$ *such that for each* $\psi \in \mathrm{End}(A)$ *satisfying* $\psi(x) \in \Gamma$, *we must have that* $B \subseteq \ker(\psi)$.

Proof. Each abelian variety is isogenous to a product of simple abelian varieties; so let $\pi : A \longrightarrow A_0 := \prod_{i=1}^r C_i^{k_i}$ be such an isogeny, where each C_i is a simple abelian variety defined over K_0. Then it suffices to find an algebraic point $y \in C := \pi(B)$ such that for each $\phi \in \mathrm{End}(A_0)$, if $\phi(y) \in \pi(\Gamma)$, then $C \subseteq \ker(\phi)$.

At the expense of replacing C with an isogenous abelian variety, we may assume that $C := \prod_{i=1}^r C_i^{m_i}$ with $0 \le m_i \le k_i$. Each endomorphism $\phi \in \mathrm{End}(A_0)$ is of the form $(J_1, \dots, J_r)$ where each $J_i \in M_{k_i}(R_i)$, where $M_{k_i}(R_i)$ is the k_i-by-k_i matrices with entries in the ring R_i of endomorphisms of C_i (note that R_i is a finite integral extension of $\mathbb{Z}$). We let Γ_i be the finitely generated R_i-module generated by the projections of $\pi(\Gamma)$ on each of the k_i copies of C_i contained in the presentation of $A_0 = \prod_{i=1}^r C_i^{k_i}$. We let $y_{i,1}, \dots, y_{i,\ell_i}$ be generators of the free part of Γ_i as an R_i-module. Without loss of generality, we may assume the points $y_{i,1}, \dots, y_{i,\ell_i}$ are linearly independent over R_i.

Then it suffices to pick $x \in C$ of the form

$$(x_{1,1}, \dots, x_{1,m_1}, x_{2,1}, \dots, x_{2,m_2}, \dots, x_{r,1}, \dots, x_{r,m_r}),$$

where each $x_{i,j} \in C_i$ such that for each i, the points $x_{i,1}, \dots, x_{i,m_i}, y_{i,1}, \dots, y_{i,\ell_i}$ are linearly independent over R_i. The existence of such points $x_{i,j}$ follows from the fact that each $C_i(\overline{K}) \otimes_{R_i} \mathrm{Frac}(R_i)$ has the structure of a $\mathrm{Frac}(R_i)$-vector space of infinite dimension. □

The next result is an application of Fact 3.8.

Lemma 5.6. *Let* K_0 *be an algebraically closed field of characteristic* 0, *let* $y_1, \dots, y_r \in A(K_0)$, *and let* $P_1, \cdots, P_r \in \mathbb{Q}[z]$ *such that* $P_i(n) \in \mathbb{Z}$ *for each* $n \ge 1$ *and for each* $i = 1, \dots, r$, *while* $\deg(P_r) > \cdots > \deg(P_1) > 0$. *For an infinite subset* $S \subseteq \mathbb{N}$, *let* $V := V(S; P_1, \dots, P_r; y_1, \dots, y_r)$ *be the Zariski closure of the set*

$$\{P_1(n)y_1 + \cdots P_r(n)y_r \colon n \in S\}.$$

Then there exist nonzero integers $\ell_1, \dots, \ell_r$ *such that* V *contains a coset of the subgroup* Γ *generated by* $\ell_1 y_1, \cdots, \ell_r y_r$.

Proof. Let Γ_0 be the subgroup of A generated by $y_1, \dots, y_r$. Because $V(K_0) \cap \Gamma_0$ is Zariski dense in V, then by Fact 3.8 we obtain that V is a finite union of cosets of algebraic subgroups of A. So, at the expense of replacing S by an infinite subset, we may assume $V = z + C$, for some $z \in A(K_0)$ and some irreducible algebraic subgroup C of A. This is equivalent with the existence of an endomorphism $\psi : A \longrightarrow A$ such that $\ker(\psi)^0 = C$ (the construction of ψ is identical with the one given in the proof of Lemma 3.7); hence ψ is constant on the set $\{P_1(n)y_1 + \cdots P_r(n)y_r\}_{n \in S}$. We will show there exist nonzero integers ℓ_i such that $\ell_i y_i \in \ker(\psi)$ for each $i = 1, \dots, r$; since $\ker(\psi)^0 = C$, then we obtain the desired conclusion.

We proceed by induction on r. The case $r = 1$ is obvious since then $\{P_1(n)\}_{n\in S}$ takes infinitely many distinct integer values (note that $\deg(P_1) \geq 1$), and so, if ψ is constant on the set $\{P_1(n)y_1\}_{n\in S}$, then $\psi(\ell y_1) = 0$ for some nonzero $\ell := P_1(n) - P_1(n_0)$ with distinct $n_0, n \in S$. Next we assume the statement holds for all $r < s$ (where $s \geq 2$), and we prove it for $r = s$.

Let $n_0 \in S$. At the expense of replacing each $P_i(n)$ by $P_i(n) - P_i(n_0)$, we may assume from now on that the set $\{P_1(n)y_1 + \cdots P_s(n)y_s\}_{n\in S}$ lies in the kernel of ψ. Let $n_1 \in S$ such that $P_1(n_1) \neq 0$ (note that $\deg(P_1) \geq 1$), and for each $i = 2, \dots, s$ we let $Q_i(z) := P_1(n_1) \cdot P_i(n) - P_1(n) \cdot P_i(n_1)$. Then the set $\{\sum_{i=2}^{s} Q_i(n)y_i\}_{n\in S}$ is in the kernel of ψ. Because $\deg(Q_i) = \deg(P_i)$ for each $i = 2, \dots, s$, we can use the induction hypothesis and conclude that there exist nonzero integers $\ell_2, \dots, \ell_s$ such that $\ell_i y_i \in \ker(\psi)$ for each i. Since $\psi(P_1(n_1)y_1 + \cdots + P_s(n_1)y_s) = 0$ and $P_1(n_1) \neq 0$, then also $\left(P_1(n_1) \cdot \prod_{i=2}^{s} \ell_i\right) y_1 \in \ker(\psi)$. This concludes our proof. ☐

Lemma 5.6 has the following important consequence for us.

Lemma 5.7. *Let K_0 be an algebraically closed field of characteristic 0, let A be an abelian variety defined over K_0, let $\tau \in \mathrm{End}(A)$ with the property that there exists a positive integer r such that $(\tau - \mathrm{id})^r = 0$, let $y \in A(K_0)$, let $\sigma : A \longrightarrow A$ be an endomorphism as algebraic varieties such that $\sigma = T_y \circ \tau$, and let $x \in A(K_0)$. Let $\gamma \in \mathrm{End}(A)$ with the property that there exists an infinite set S of positive integers such that γ is constant on the set $\{\sigma^n(x) \colon n \in S\}$. Then there exists a positive integer ℓ such that $\ell \cdot (\beta(x) + y) \in \ker(\gamma)$, where $\beta := \tau - \mathrm{id}$.*

Proof. We compute $\sigma^n(x)$ for any $n \in \mathbb{N}$; first of all, we have

$$\sigma^n(x) = \tau^n(x) + \sum_{i=0}^{n-1} \tau^i(y). \tag{5.7.1}$$

Then (since $\beta = \tau - \mathrm{id}$ and also) noting that $\beta^r = 0$ we have

$$\sigma^n(x) \tag{5.7.2}$$

$$= \sum_{i=0}^{n} \binom{n}{i} \beta^i(x) + \sum_{i=0}^{n-1} \tau^i(y) \tag{5.7.3}$$

$$= \sum_{i=0}^{r-1} \binom{n}{i} \beta^i(x) + \sum_{i=0}^{n-1} \sum_{j=0}^{i} \binom{i}{j} \beta^j(y) \tag{5.7.4}$$

$$= \sum_{j=0}^{r-1} \binom{n}{j} \beta^j(x) + \sum_{j=0}^{r-1} \left(\sum_{i=j}^{n-1} \binom{i}{j} \right) \beta^j(y) \tag{5.7.5}$$

$$= \sum_{j=0}^{r-1} \binom{n}{j} \beta^j(x) + \sum_{j=0}^{r-1} \binom{n}{j+1} \beta^j(y) \tag{5.7.6}$$

$$= x + \sum_{j=1}^{r} \binom{n}{j} \beta^j(x) + \sum_{j=1}^{r} \binom{n}{j} \beta^{j-1}(y) \tag{5.7.7}$$

$$= x + \sum_{j=1}^{r} \binom{n}{j} \beta^{j-1} \left(\beta(x) + y\right). \tag{5.7.8}$$

Since γ is constant on the set $\{\sigma^n(x) \colon n \in S\}$, then letting $n_1 \in S$ we have that for each $n \in S$,

$$\sum_{j=1}^{r} \left(\binom{n}{j} - \binom{n_1}{j} \right) \beta^{j-1} \left(\beta(x) + y\right) \in \ker(\gamma). \tag{5.7.9}$$

Using Lemma 5.6 and (5.7.9), we obtain the desired conclusion. □

Then the following result is an immediate consequence of Lemma 5.7 and of Lemma 3.9.

Corollary 5.8. *With the notation as in Lemma 5.7, if the cyclic group generated by $\beta(x) + y$ is Zariski dense in A, then $\gamma = 0$. Moreover, the set $\{\sigma^n(x) \colon n \in S\}$ is Zariski dense in A.*

Proof. Indeed, Lemmas 3.9 and 5.7 yield that any group homomorphism γ which is constant on the set $U := \{\sigma^n(x) \colon n \in S\}$ must be trivial.

Now, for the 'moreover' part of Corollary 5.8, Fact 3.6 yields that U (along with $\mathcal{O}_\sigma(x)$) is contained in a finitely generated subgroup of A, and so, Fact 3.8 yields that the Zariski closure of U is a finite union of cosets of algebraic subgroups of A. Pick such a coset $w + H$ which contains infinitely many $\sigma^n(x)$. Then another application of Lemma 5.7 (coupled with Lemmas 3.7 and 3.9) yields that $H = A$, thus completing our proof that U is Zariski dense in A. □

6. The cyclic case

Now we are ready to prove Theorem 1.3 for cyclic monoids.

Proof of Theorem 1.2. Let K be a finitely generated subfield of K_0 such that both A and σ are defined over K. Let $\overline{K}$ be the algebraic closure of K inside K_0; clearly, it suffices to prove Theorem 1.2 with K_0 replaced by $\overline{K}$.

By Fact 3.4, there exists an isogeny $\tau : A \longrightarrow A$, and there exists $y \in A(K)$, such that $\sigma(x) = \tau(x) + y$ for all $x \in A$. At the expense of replacing σ by an iterate σ^n (and in particular, replacing τ by τ^n; see also (5.7.1)), we may assume $\dim\ker(\tau^m - \mathrm{id}) = \dim(\ker(\tau - \mathrm{id}))$ for all $m \in \mathbb{N}$ (see Lemma 4.1 which shows that it is sufficient to prove Theorem 1.2 for an iterate of σ). In other words, we may assume that the only root of unity, if any, which is a root of the minimal polynomial f (with coefficients in $\mathbb{Z}$) of $\tau \in \mathrm{End}(A)$ is equal to 1.

Let r be the order of vanishing at 1 of f, and let $f_1 \in \mathbb{Z}[t]$ such that $f(t) = f_1(t)\cdot(t-1)^r$. Then f_1 is also a monic polynomial, and if $r = 0$, then $f_1 = f$. Let $A_1 := (\tau - \mathrm{id})^r(A)$ and let $A_2 := f_1(\tau)(A)$, where $f_1(\tau) \in \mathrm{End}(A)$ and id is the identity map on A. If $r = 0$, then $A_2 = 0$ and therefore $A_1 = A$. By definition, both A_1 and A_2 are connected algebraic subgroups of A, hence they are both abelian subvarieties of A. Furthermore, by definition, the restriction of $\tau|_{A_1} \in \mathrm{End}(A_1)$ has minimal polynomial equal to f_1 whose roots are not roots of unity. On the other hand, $(\tau - \mathrm{id})^r|_{A_2} = 0$.

Lemma 6.1. *With the above notation, $A = A_1 + A_2$ and $A_1 \cap A_2$ is finite.*

Proof of Lemma 6.1. By the definition of r and of f_1, we know that the polynomials $f_1(t)$ and $(t-1)^r$ are coprime; so there exist polynomials $g_1, g_2 \in \mathbb{Z}[t]$ and there exists a nonzero integer k (the resultant of $f_1(t)$ and of $(t-1)^r$) such that

$$f_1(t) \cdot g_1(t) + (t-1)^r \cdot g_2(t) = k.$$

Let $x \in A(\overline{K})$ and let $x_0 \in A(\overline{K})$ such that $kx_0 = x$. Then clearly

$$x_1 := (\tau - \mathrm{id})^r\,(g_2(\tau)x_0) \in A_1 \text{ and } x_2 := f_1(\tau)\,(g_1(\tau)x_0) \in A_2,$$

and moreover, $x_1 + x_2 = kx_0 = x$, as desired.

Arguing similarly, one can show that $A_1 \cap A_2 \subseteq A[k]$ since if $x \in A_1 \cap A_2$ then $f_1(\tau)x = 0 = (\tau - \mathrm{id})^r x$ and thus

$$kx = (g_1(\tau)f_1(\tau) + g_2(\tau)(\tau - \mathrm{id})^r)\,x = 0,$$

as desired. □

Let $y_1 \in A_1$ and $y_2 \in A_2$ such that $y = y_1 + y_2$; furthermore, we may assume that if $y_1 \in A_2$ then $y_1 = 0$. We note that τ restricts to an endomorphism to each A_1 and A_2; we denote by τ_i the action of τ on each A_i. Let $y_0 \in A_1(\overline{K})$ such that $(\mathrm{id} - \tau_1)(y_0) = y_1$ (note that $(\mathrm{id} - \tau_1) : A_1 \longrightarrow A_1$ is an isogeny because the minimal polynomial f_1 of $\tau_1 \in \mathrm{End}(A_1)$ does not have the root 1). Using Lemma 5.4, it suffices to prove Theorem 1.2 for $T_{-y_0} \circ \sigma \circ T_{y_0}$; so, we may and do assume that $y_1 = 0$.

Let $\sigma_i : A_i \longrightarrow A_i$ be given by $\sigma_1(x) = \tau_1(x)$ and $\sigma_2(x) = \tau_2(x) + y_2$. Then for each $x \in A$, we let $x_1 \in A_1$ and $x_2 \in A_2$ such that $x = x_1 + x_2$; we have:

$$\sigma(x) = \sigma(x_1 + x_2) = \tau(x_1 + x_2) + y_2 = \tau(x_1) + \tau(x_2) + y_2 = \sigma_1(x_1) + \sigma_2(x_2).$$

Moreover, $\sigma^n(x_1 + x_2) = \sigma_1^n(x_1) + \sigma_2^n(x_2)$ for all $n \in \mathbb{N}$.

We let $\beta := (\tau_2 - \mathrm{id})|_{A_2} \in \mathrm{End}(A_2)$; then $\beta^r = 0$. Let B be the Zariski closure of the subgroup of A_2 generated by $\beta(A_2)$ and y_2. Then B is an algebraic subgroup of A_2.

Lemma 6.2. *If $B \neq A_2$, then σ preserves a nonconstant fibration.*

Proof of Lemma 6.2. If $B \neq A_2$, then $\dim(B) < \dim(A_2)$ (note that A_2 is connected) and since $A_2 \cap A_1$ is finite, we conclude that the algebraic subgroup $C := A_1 + B$ is a proper abelian subvariety of A. We let $f : A \longrightarrow A/C$ be the quotient map; we claim that $f \circ \sigma = f$. Indeed, for each $x \in A$, we let $x_1 \in A_1$ and $x_2 \in A_2$ such that $x = x_1 + x_2$ and then

$$f(\sigma(x)) = f(\sigma(x_1 + x_2)) = f(\sigma_1(x_1) + \sigma_2(x_2)) = f(\sigma_2(x_2)) = f(x_2) = f(x).$$

Since A/C is a positive dimensional algebraic group and $f : A \longrightarrow A/C$ is the quotient map, then we conclude that σ preserves a nonconstant fibration. $\square$

From now on, assume $B = A_2$. We will prove that there exists $x \in A(\overline{K})$ such that $\mathcal{O}_\sigma(x)$ is Zariski dense in A. First we prove there exists $x_2 \in A_2(\overline{K})$ such that $\mathcal{O}_{\sigma_2}(x_2)$ is Zariski dense in A_2.

Because we assumed that the group generated by $\beta(A_2)$ and y_2 is Zariski dense in A, then Lemma 5.1 yields the existence of $x_2 \in A_2(\overline{K})$ such that the group generated by $\beta(x_2) + y_2$ is Zariski dense in A_2. Then Corollary 5.8 yields that any infinite subset of $\mathcal{O}_{\sigma_2}(x_2)$ is Zariski dense in A_2. If A_1 is trivial, then $A_2 = A$ and $\sigma_2 = \sigma$ and Theorem 1.2 is proven. So, from now on, assume that A_1 is positive dimensional.

Let Γ be the subgroup of $A(\overline{K})$ generated by all $\phi(x_2)$ and $\phi(y_2)$ as we vary $\phi \in \mathrm{End}(A)$. Then Γ is a finitely generated $\mathrm{End}(A)$-module. Using Lemma 5.5, we may find $x_1 \in A_1(\overline{K})$ with the property that if $\psi \in \mathrm{End}(A)$ has the property that $\psi(x_1) \in \Gamma$, then $A_1 \subseteq \ker(\psi)$. Let $x := x_1 + x_2$; we will prove that $\mathcal{O}_\sigma(x)$ is Zariski dense in A.

Let V be the Zariski closure of $\mathcal{O}_\sigma(x)$. The orbit $\mathcal{O}_\sigma(x)$ is contained in a finitely generated group (see Fact 3.6). Then Fact 3.8 yields that V is a finite union of cosets of algebraic subgroups of A. So, if $V \neq A$, then there exists a coset $c + C$ of a proper algebraic subgroup $C \subset A$ which contains $\{\sigma^n(x)\}_{n \in S}$ for some infinite subset $S \subseteq \mathbb{N}$. By Lemma 3.7, there exists a nonzero $\psi \in \mathrm{End}(A)$ such that $\psi(\sigma^n(x)) = \psi(c)$ for each $n \in S$, i.e. ψ is constant on the set $\{\sigma^n(x) \colon n \in S\}$.

Let $n > m$ be two elements of S. Then $\psi(\sigma^n(x) - \sigma^m(x)) = 0$, and so,

$$\psi(\tau_1^n - \tau_1^m)(x_1) = \psi(\sigma_2^m - \sigma_2^n)(x_2) \in \Gamma.$$

Using the fact that $x_1 \in A_1$ was chosen to satisfy the conclusion of Lemma 5.5 with respect to Γ and the fact that $\tau_1^n - \tau_1^m = \tau_1^m(\tau_1^{n-m} - \mathrm{id})$ is an isogeny on A_1, we obtain that $\psi(A_1) = 0$. Thus ψ is constant on $\{\sigma_2^n(x_2)\}_{n\in S}$. Then Corollary 5.8 yields that $A_2 \subseteq \ker(\psi)$. Hence $A_1 + A_2 = A \subseteq \ker(\psi)$ which contradicts the fact that $\psi \neq 0$. This concludes our proof. $\square$

7. The general case

The proof of Theorem 1.3 follows the same strategy as the proof of Theorem 1.2.

Proof of Theorem 1.3. We let $\gamma_1, \dots, \gamma_s$ be a set of generators for S. As before, we let K be a finitely generated subfield of K_0 such that A and each γ_i are defined over K. Also, we may (and do) assume that K_0 is a given algebraic closure $\overline{K}$ of K.

We let S_0 be the monoid of group endomorphisms of A consisting of all $\tau : A \longrightarrow A$ such that there exists some $y \in A$ such that $T_y \circ \tau \in S$. We let $\mathcal{U} := \{\gamma_1, \dots, \gamma_s\}$, and also let $\mathcal{U}_0$ be a finite set of generators for S_0 corresponding to the elements in $\mathcal{U}$ (i.e., for each $\varphi \in \mathcal{U}_0$, there exists $y \in A$ such that $T_y \circ \varphi \in \mathcal{U}$).

By Fact 3.2, A is isogenuous with a product of simple abelian varieties $\prod_i A_i^{r_i}$ and so, $\mathrm{End}(A)$ (the ring of group endomorphisms of A) is isomorphic to $\prod_i M_{r_i}(\mathrm{End}(A_i))$. We let $R_i := \mathrm{End}(A_i)$ and $F_i := \mathrm{Frac}(R_i)$. Then each element in S_0 is represented by a tuple of matrices in $\prod_i M_{r_i}(R_i)$; from now on, we use freely this identification of the group endomorphisms from S_0 with tuples of matrices in $\prod_i M_{r_i}(R_i)$. Using Lemma 2.5 and also Lemma 4.1, it suffices to assume that for each $\tau \in S_0$, and for each positive integer n, we have

$$\dim \ker(\tau - \mathrm{id}) = \dim \ker(\tau^n - \mathrm{id}). \tag{7.0.1}$$

Let U_0 be the submonoid of S_0 generated by all $\tau \in S_0$ such that

$$\max_{n\geq 1} \dim \ker(\tau - \mathrm{id})^n \tag{7.0.2}$$

is minimal as we vary τ in S_0. Then, by Lemma 2.6, $\bar{U_0} = S_0$. Let U be the submonoid of S corresponding to U_0, i.e. the set of all $\sigma \in S$ such that there exists some $\tau \in U_0$ and there exists a translation T_y on A for which $\sigma = T_y \circ \tau$. Because $\bar{U_0} = S_0$, then also $\bar{U} = S$. Using Lemma 2.3, there exists a finitely generated submonoid U' of U (and therefore of S) and there exists a positive integer n such that for each $i = 1, \dots, s$, we have $\gamma_i^n \in \bar{U}'$. By Lemma 4.3, it suffices to prove Theorem 1.3 for U'. So, from now on, we assume $U' = S$. In particular, this means that S_0 is generated (as a monoid) by finitely many endomorphisms τ satisfying (7.0.2); we denote this set by $\mathcal{U}_0$ (as before). Finally, we recall our notation that $\mathcal{U} = \{\gamma_1, \dots, \gamma_s\}$ is a finite set of generators of S, and that for each generator $\tau \in \mathcal{U}_0$ of S_0 there exists some translation T_y and some $i = 1, \dots, s$ such that $T_y \circ \tau = \gamma_i$.

Let τ_1, τ_2 in $\mathcal{U}_0$. Assume r_1 is the order of the root 1 of the minimal polynomial for τ_1, and let $B_2 := \ker(\tau_1 - \mathrm{id})^{r_1}$. Since τ_2 commutes with τ_1, we obtain that τ_2 acts on B_2. Furthermore, because both τ_1 and τ_2 are in $\mathcal{U}_0$, it must be that the restriction of the action of τ_2 on B_2 is also unipotent (see also the proof of Lemma 2.6); otherwise for some positive integer m, the element $\tau := \tau_2^m \tau_1 \in S_0$ would have the property that

$$\max_{n \geq 1} \dim \ker(\tau - \mathrm{id})^n$$

is smaller than $\dim B_2$ (which is minimal among all elements of S_0).

We let B_1 be a complementary connected algebraic subgroup of A such that $A = B_1 + B_2$, and moreover, each element of S induces an endomorphism of B_1. So, we reduced to the case that each element of S is of the form $T_y \circ \tau$, where τ acts on $A = B_1 + B_2$ as follows:

(i) τ restricted to B_2 acts unipotently, i.e. there exists some positive integer r_τ such that $(\tau - \mathrm{id})^{r_\tau}|_{B_2} = 0$;

(ii) for each $\tau \in \mathcal{U}_0$, the action of τ on B_1 (which by abuse of notation, we also denote by τ) has the property that $\tau^n - \mathrm{id}$ is a dominant map for each positive integer n (see (7.0.1)).

We proceed similarly to the case S is cyclic. Then for each $\sigma_i \in \mathcal{U}$ (for $i = 1, \dots, s$), we let $\tau_i \in \mathcal{U}_0$, $z_i \in B_1$ and $y_i \in B_2$ such that $\sigma_i = T_{y_i + z_i} \circ \tau_i$. Note that it may be that $\tau_i = \tau_j$ for some $i \neq j$, but this is not relevant for the proof. We let C_i be the algebraic subgroup of B_2 spanned by y_i and $(\tau_i - \mathrm{id})(B_2)$ (for each $i = 1, \dots, s$). We recall that $\beta_i := (\tau_i - \mathrm{id})|_{B_2}$ is a nilpotent endomorphism of B_2; we let $\mathcal{U}_1$ be the finite set of all β_i. Finally, we let C_S be the algebraic subgroup of B_2 generated by all C_i.

If the algebraic subgroup $C_S + B_1$ does not equal A, then the exact same argument as in Lemma 6.2 yields the existence of a non-constant rational map fixed by each $\sigma \in S$. Essentially, the projection map $\pi : A \longrightarrow A/(B_1 + C_S)$ is a non-constant morphism with the property that $\pi \circ \sigma = \pi$ for each $\sigma \in S$.

Next assume $C_S + B_1 = A$; we will show there exists $x \in A(\overline{K})$ whose orbit under S is Zariski dense. The strategy is the same as in the case S is cyclic. We can find algebraic points $x_1 \in B_1$ and $x_2 \in B_2$ such that the S-orbit of $x = x_1 + x_2$ is Zariski dense in A. First we choose $x_2 \in B_2(\overline{K})$ as in Lemma 5.1 with respect to the algebraic group endomorphisms β_i and the points y_i, for $i = 1, \dots, s$; hence the Zariski closure of the group generated by $\beta_i(x_2) + y_i$ is C_i for each i.

Let Γ be the $\mathrm{End}(A)$-module spanned by $x_2, y_1, \dots, y_s, z_1, \dots, z_s$, which is a finitely generated subgroup of $A(\overline{K})$. Then (using Lemma 5.5) we choose $x_1 \in B_1(\overline{K})$ such that if $\psi \in \mathrm{End}(A)$ has the property that $\psi(x_1) \in \Gamma$, then $B_1 \subseteq \ker(\psi)$. Let $x := x_1 + x_2$; we will prove that $\mathcal{O}_S(x)$ is Zariski dense in A.

Using Facts 3.6 and 3.8, the Zariski closure of $\mathcal{O}_S(x)$ is a union of finitely many cosets $w_j + H_j$ of algebraic subgroups of A.

Lemma 7.1. *There exists a coset $w+H$ of an algebraic subgroup appearing as a component of the Zariski closure of $\mathcal{O}_S(x)$, and there exists a positive integer N such that $w+H$ is invariant under γ_i^N for each $i=1,\dots,s$.*

Proof. So, we know that the Zariski closure of $\mathcal{O}_S(x)$ is the union of cosets of (irreducible) algebraic subgroups $\cup_{i=1}^{\ell}(w_i+H_i)$. Let $\gamma\in S$. Then, using that $\gamma\left(\mathcal{O}_S(x)\right)\subseteq\mathcal{O}_S(x)$, we obtain

$$\cup_{i=1}^{\ell}\left(\gamma(w_i)+\gamma(H_i)\right)\subseteq\cup_{i=1}^{\ell}(w_i+H_i).$$

On the other hand, each $\gamma\in S$ is a dominant endomorphism of A, and therefore, for each $i=1,\dots,\ell$, we have $\dim(\gamma(H_i))=\dim(H_i)$. So, that means γ permutes the subgroups H_i of maximal dimension appearing above. In particular, there exists a positive integer N_0 such that for each $i=1,\dots,s$, the endomorphism $\gamma_i^{N_0}$ fixes each algebraic group H_i of maximal dimension.

Let $S^{(N_0)}$ be the submonoid of S consisting of all γ^{N_0} for $\gamma\in S$. Now, let H be one such algebraic group of maximal dimension among the algebraic groups H_i (for $i=1,\dots,\ell$). Let w_i+H with $i=1,\dots,k$ be all the cosets of H appearing as irreducible components of the Zariski closure of $\mathcal{O}_S(x)$. Then each element $\gamma\in S^{(N_0)}$ induces a map $f_\gamma:\{1,\dots,k\}\longrightarrow\{1,\dots,k\}$ given by $f_\gamma(w_i+H)=w_{f_\gamma(i)}+H$; the map is not necessarily bijective. Moreover, we get a homomorphism of monoids $f:S^{(N_0)}\longrightarrow F_k$ given by $f(\gamma):=f_\gamma$, where F_k is the monoid of all functions from the set $\{1,\dots,k\}$ into itself. Clearly, there exists $j\in\{1,\dots,k\}$, and there exists a positive integer N_1 such that $f_{\gamma^{N_1}}(j)=j$ for each generator $\gamma\in\{\gamma_1^{N_0},\dots,\gamma_s^{N_0}\}$ of $S^{(N_0)}$. Then Lemma 7.1 holds with $N:=N_0\cdot N_1$. $\square$

Let $w+H$ be one coset as in the conclusion of Lemma 7.1, and let N be the positive integer from the conclusion of Lemma 7.1 with respect to the coset $w+H$. We let S' be the submonoid of S generated by γ_i^N for $i=1,\dots,s$. Then $w+H$ contains a set of the form $\mathcal{O}_{S'}(x')$, for some $x'\in\mathcal{O}_S(x)$; in other words, $w+H$ contains a set of the form

$$\{\gamma_1^{m_1+Nn_1}\cdots\gamma_s^{m_s+Nn_s}(x)\,:\,n_1,\dots,n_s\geq 0\},$$

for some positive integers $m_1,\dots,m_s$.

Let then $\pi:A\longrightarrow A/H$ be the canonical projection. Then

$$\pi\left(\gamma_1^{m_1+Nn_1}\cdots\gamma_s^{m_s+Nn_s}(x)\right)=w$$

for all $n_1,\dots,n_s\geq 0$. Restricted on B_1, for each group endomorphism τ_i (for $i=1,\dots,s$), the action on the tangent space of B_1 has no eigenvalue which is a root of unity (see (ii) above); hence

$$\psi_1:=\left(\tau_1^{m_1+N}\tau_2^{m_2}\cdots\tau_s^{m_s}-\tau_1^{m_1}\tau_2^{m_2}\cdots\tau_s^{m_s}\right)|_{B_1}\text{ is an isogeny.}$$

So, we get that $(\pi\circ\psi_1)(x_1)\in\Gamma$. Because of our choice for x_1 and the fact that ψ_1 is an isogeny on B_1, we conclude that $B_1\subseteq\ker(\pi)$ (note also that B_1 is connected by our assumption). Thus $B_1\subseteq H$. So, we can view π as a group homomorphism $\pi : B_2 \longrightarrow A/H$ with the property that for each $n_1, \dots, n_s \geq 0$ we have

$$\pi\left(\gamma_1^{m_1+n_1N}\cdots\gamma_s^{m_s+n_sN} - \gamma_1^{m_1}\cdots\gamma_s^{m_s}\right)(x_2) = 0.$$

Letting $\gamma' := \gamma_1^{m_1}\cdots\gamma_s^{m_s}|_{B_2}$, we have that $\pi\circ\gamma'$ is constant (equal to w) on each orbit $\mathcal{O}_{\gamma_i^N}(x_2)$. Then Corollary 5.8 yields that the connected component of the Zariski closure C_i of the cyclic group generated by $(\tau_i - \mathrm{id})(x_2) + y_i$ is contained in the kernel of $\pi\circ\gamma'$. Since the $C_i's$ generate the algebraic group C_S (and therefore the connected components of the C_i's generate the connected component of C_S; see also Fact 3.1), and furthermore, the connected component of C_S contains the connected component of B_2, we conclude that $\pi\circ\gamma'$ is identically 0 on B_2. Because γ' is an isogeny, we conclude that $B_2 \subseteq \ker(\pi)$, and therefore $H = A$ since H contains both B_1 and B_2. This concludes our proof. □

References

[ABR11] E. Amerik, F. Bogomolov, and M. Rovinsky, *Remarks on endomorphisms and rational points*, Compos. Math. **147** (2011), 1819–1842.

[AC08] E. Amerik and F. Campana, *Fibrations méromorphes sur certaines variétè a fibré canonique trivial*, Pure Appl. Math. Q. **4** (2008), 509–545.

[BGT] J. P. Bell, D. Ghioca, and T. J. Tucker, *Applications of p-adic analysis for bounding periods of subvarieties under etale maps*, to appear in Int. Math. Res. Not. (2014), 18 pages.

[BGZ] J. P. Bell, D. Ghioca, and Z. Reichstein, *On a dynamical version of a theorem of Rosenlicht*, preprint (2014), 15 pages, `arXiv:1408.4744`.

[BRS10] J. P. Bell, D. Rogalski, and S. J. Sierra, *The Dixmier-Moeglin equivalence for twisted homogeneous coordinate rings*, Israel J. Math. **180** (2010), 461–507.

[CS93] G. S. Call and J. H. Silverman, *Canonical heights on varieties with morphisms*, Compositio Math. **89** (1993), 163–205.

[Fal94] G. Faltings, *The general case of S. Lang's conjecture*, Barsotti Symposium in Algebraic Geometry (Abano Terme, 1991), Perspect. Math., no. 15, Academic Press, San Diego, CA, 1994, pp. 175–182.

[FJ08] M. D. Fried and J. Moshe, *Field arithmetic*. Third edition. Springer-Verlag, Berlin, 2008. xxiv+792 pp.

[Kol48] E. R., Kolchin, *Algebraic matric groups and the Picard-Vessiot theory of homogeneous linear ordinary differential equations*, Ann. of Math. (2) **49**, (1948), 1 - 42.

[MS14] A. Medvedev and T. Scanlon, *Invariant varieties for polynomial dynamical systems*, Ann. of Math. (2) **179** (2014), no. 1, 81–177.

[Mil] J. Milne, *Abelian varieties*, course notes available online: `http://www.jmilne.org/math/CourseNotes/av.html`.

[Mum70] D. Mumford, *Abelian varieties.* Tata Institute of Fundamental Research Studies in Mathematics **5**, published for the Tata Institute of Fundamental Research, Bombay; Oxford University Press, London 1970 viii+242 pp.

[Tho13] C. Thornhill, *Abelian varieties and Galois extensions of Hilbertian fields*, J. Inst. Math. Jussieu **12** (2013), no. 2, 237–247.

[Zha10] S. W. Zhang, *Distributions in algebraic dynamics*, a tribute to professor S. S. Chern. volume 10 of Survey in Differential Geometry, pages 381–430. International Press, 2006.

Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada
E-mail address: dghioca@math.ubc.ca

Mathematics Department, University of California Berkeley, Evans Hall, Berkeley CA, 94720-3840
E-mail address: scanlon@math.berkeley.edu