

# Differential algebraic Zilber-Pink theorems

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## Whose work is this?

- I will be discussing some work with Jonathan Pila, which is still in progress. The theorems described today are those which have been worked out in detail; the full scope of what we might do with these methods is not yet clear.
- Vahagn Aslanyan has proven similar theorems (see especially his preprint “Weak modular Zilber-Pink with derivatives”, [arXiv:1803.05895](https://arxiv.org/abs/1803.05895)) and will speak about his work on Friday. My sense is that our methods and results overlap in that we are exploiting the uniformities implicit in the differential algebraic formulation of Ax-Schanuel but that we diverge in our use of these uniformities, in our approach to the existential closedness conditions, and in our focus on algebraic versus differential algebraic Zilber-Pink.

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# The context

- Our results apply to Shimura varieties and more generally to variations of Hodge structures as in the work of Bakker, Klingler and Tsimerman,
- I presume that our methods generalize to mixed Shimura varieties and more generally variations of mixed Hodge structures, but there are some technical points we have not considered in detail (and the requisite Ax-Schanuel theorems are, to my knowledge, not yet known).
- With the aim of making this presentation more elementary, I will restrict attention to the case where the ambient space is  $\mathbb{A}^k = Y_0(1)^k$  regarded as the moduli space of products of  $k$  elliptic curves.

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## Special subvarieties

We let  $\mathfrak{H} := \{\tau \in \mathbb{C} : \text{Im}(\tau) > 0\}$  be the upper half plane and normalize the analytic  $j$ -function  $j : \mathfrak{H} \rightarrow \mathbb{C}$  by  $j(i) = 1728$  and  $j(\exp(2\pi i/3)) = 0$ . For  $N \in \mathbb{Z}_+$ , we denote by  $\Phi_N(x, y) \in \mathbb{Z}[x, y]$  the irreducible polynomial for which  $\Phi_N(j(\tau), j(N\tau)) \equiv 0$ .

A subvariety of  $\mathbb{A}^k$  which is a component of a variety defined by equations of the form  $\Phi_N(x_i, x_j) = 0$  for various  $1 \leq i \leq j \leq k$  is called **special**. A component of a variety which is defined by such equations and possibly also equations of the form  $x_\ell = \xi$  for various  $\ell \leq k$  and parameters  $\xi$  is called **weakly special**. A special variety for which no coordinate projection restricts to a constant map is called **strongly special**.

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# Special loci

For each pair of natural numbers  $r \leq k$ , let  $\mathcal{S}^{[r,k]} = \mathcal{S}^{[r]}$  be the union of all special subvarieties of  $\mathbb{A}^k$  of dimension  $r$ .

We write  $\mathcal{SS}^{[r,k]} = \mathcal{SS}^{[r]}$  for the union of the **strongly** special subvarieties of  $\mathbb{A}^k$  of dimension  $r$ .

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## Unlikely intersections

If  $B \subseteq \mathbb{A}^k$ , then  $\langle B \rangle$  is the smallest special subvariety of  $\mathbb{A}^k$  containing  $B$  and  $\langle B \rangle_{\text{ws}}$  is the smallest weakly special variety containing  $B$ .

Let  $Z \subseteq \mathbb{A}^k$  be an irreducible subvariety of  $\mathbb{A}^k$  and  $r := \dim \langle Z \rangle_{\text{ws}} - \dim Z - 1$ . The **unlikely locus** in  $Z$  is

$$\text{Unl}(Z) := Z \cap \mathcal{S}^{[r,k]}$$

whereas the **strongly unlikely locus** in  $Z$  is

$$\text{sUnl}(Z) := Z \cap \mathcal{SS}^{[r,k]}$$

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# Zilber-Pink in the form of unlikely intersections

The Zilber-Pink conjectures admit variants. A natural form for subvarieties of  $\mathbb{A}^k$  says:

## Conjecture

If  $Z \subseteq \mathbb{A}^k$  is an irreducible subvariety, then  $\overline{\text{Unl}(Z)}^{\text{Zariski}} \neq Z$ .

- In this form, the Zilber-Pink conjecture includes André-Oort for  $Y_0(1)^k$ .
- Habegger and Pila have reduced this conjecture to a statement about Galois actions on zero dimensional unlikely intersections.
- We shall consider a differential algebraic approach which requires us to ignore constant points.

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## Weak Zilber-Pink

Let  $K \supseteq \mathbb{C}$  be some field extending the complex numbers. For any subset  $B \subseteq \mathbb{A}^k(K)$ , we set  $B^\dagger := B \cap (K \setminus \mathbb{C})^k$ .

Possibly enlarging  $K$ , we endow  $K$  with a derivation  $\partial : K \rightarrow K$  so that  $\ker \partial = \mathbb{C}$  and  $(K, \partial)$  is a saturated differentially closed field.

### Theorem

*Let  $Z \subseteq \mathbb{A}_K^k$  be an irreducible variety, then  $\overline{\text{Unl}(Z)^\dagger}^{\text{Zariski}} \neq Z$ . Moreover,  $\deg(\overline{\text{Unl}(Z)^\dagger}^{\text{Zariski}})$  may be bounded by a function of the form  $C_k \deg(Z)^k$ .*

- The constant  $C_k$  can be made explicit by computing the volumes of certain Newton polyhedra using the work of Binyamini.
- The qualitative form of this theorem differs from earlier weak modular Zilber-Pink theorems in that we are also controlling zero dimensional atypical intersections.
- The nondensity of  $\text{Unl}(Z)^\dagger$  in  $Z$  (where  $B^\dagger$  is suitably redefined) holds for subvarieties of Shimura varieties and more generally for VHSs, but we do not have the degree bounds.

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## Theorem (Pila and Tsimerman)

Let  $U \subseteq \mathbb{C}^n$  be an open domain and  $f = (f_1, \dots, f_k) : U \rightarrow \mathfrak{H}^k$  an analytic function for which no geodesic relation holds (that is, no  $f_i$  is constant and for no  $i \neq j$  do we have  $f_i = \gamma \cdot f_j$  for some  $\gamma \in \mathrm{GL}_2(\mathbb{Q})$ ). Then

$$\mathrm{tr. \, deg}_{\mathbb{C}}(\mathbb{C}(f_1, \dots, f_k, j(f_1), \dots, j(f_k))) \geq k + \mathrm{rk}\left(\frac{\partial f_i}{\partial z_j}\right).$$

- Extensions of this Ax-Schanuel theorem have been proven for general coverings of Shimura varieties by Mok-Pila-Tsimerman and for period mappings associated to VHS by Bakker-Tsimerman. Our methods allow for a deduction of weak Zilber-Pink from these.
- This theorem has a differential algebraic form and then extensions (due to Pila and then Mok-Pila-Tsimerman) taking into account differential equations.

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## Differential equations associated to the $j$ -function

The Schwarzian derivative with respect to the derivation  $\partial$  is the differential rational function

$$S(x) = S_{\partial}(x) := \partial\left(\frac{\partial^2 x}{\partial x}\right) - \frac{1}{2}\left(\frac{\partial^2 x}{\partial x}\right)^2.$$

The logarithmic derivative associated to the  $j$ -function is the differential operator

$$\chi(x) := S(x) + (\partial x)^2 \frac{x^2 - 1968x + 2654208}{2x^2(x - 1728)^2}.$$

If  $U \subseteq \mathbb{C}$  is an open domain and  $f, g : U \rightarrow \mathfrak{H}$  are nonconstant analytic functions, then

$$\chi(j(f)) = \chi(j(g)) \iff S(f) = S(g) \iff (\exists \gamma \in \mathrm{GL}_2(\mathbb{C})) \gamma \cdot f = g$$

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## Differential equations for special loci

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For  $r \leq k$ , let

$$\Xi^{[r,k]} = \Xi^{[r]} := \bigcup_{\substack{\Pi \text{ a partition of } \{1, \dots, k\} \\ \#\Pi = r}} \Xi_{\Pi}$$

- Observe that  $(\mathcal{S} \mathcal{S}^{[r,k]})^{\dagger} \subseteq \Xi^{[r,k]}$ .
- Similar algebraic differential equations cut out the union of the special loci of a given dimension for Shimura varieties and VHS.

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# From Ax-Schanuel to weak Zilber-Pink

- We shall show that if  $Z \subseteq \mathbb{A}^k$  is irreducible,  $\ell := \dim\langle Z \rangle_{\text{ws}}$  and  $r := \ell - \dim(Z) - 1$ , then  $Z \cap \Xi^{[k,r]}$  is contained in the union of finitely many proper algebraic subvarieties of  $Z$  obtained as intersections with strongly specials.
- If  $Z \cap \Xi^{[r]}$  were Zariski dense in  $Z$ , then we could find a differential component  $U$  of the intersection which is Zariski dense in  $Z$ .
- Let  $L$  be a field finitely generated over  $\mathbb{C}$  over which  $Z$  is defined.
- Let  $(u_i = u_{i,1}, \dots, u_{i,k})_{i=1}^{\infty}$  be a **Morley sequence** in  $U$ .
- Using the Seidenberg embedding theorem, let  $(\tilde{u}_i = \tilde{u}_{i,1}, \dots, \tilde{u}_{i,k})_{i=1}^{\infty}$  be a sequence of germs of meromorphic functions so that we realize  $u_{i,j} = j(\tilde{u}_{i,j})$ .

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- Let  $L$  be a field finitely generated over  $\mathbb{C}$  over which  $Z$  is defined.
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# The contradiction

- From our hypothesis on  $U$ , there are no more weakly special relations on  $U$  than on  $Z$ . Thus, for any natural number  $N$ , the Ax-Schanuel theorem yields that

$$\text{tr. deg}_{\mathbb{C}} \mathbb{C}(\tilde{u}_1, \dots, \tilde{u}_N, u_1, \dots, u_N) \geq N \cdot \ell + 1$$

- Since  $U \subseteq \Xi^{[r]}$ , there are at most  $r$  geodesically independent entries in  $(\tilde{u}_{i,1}, \dots, \tilde{u}_{i,k})$ .
- Thus, we compute an upper bound:

$$\begin{aligned} \text{tr. deg}_{\mathbb{C}} L(\tilde{u}_1, \dots, \tilde{u}_N, u_1, \dots, u_N) &\leq \text{tr. deg}_{\mathbb{C}}(L) + rN + \dim(Z)N \\ &= \text{tr. deg}_{\mathbb{C}}(L) + (r + \dim(Z))N \end{aligned}$$

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# Effective bounds in Zilber-Pink

- There are general theorems which give bounds on the degree of the Zariski closure of a differential algebraic variety as a function of combinatorial data associated to the defining differential equations.
- In the case of  $Z \cap \Xi^{[r]}$ , the simpler methods of Hrushovski and Pillay would give a bound of  $6^{(k-r)(8^k-1)} \deg(Z)^{8^k-1}$ .
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## Likely density

In order to convert these bounds on  $\deg(\overline{Z \cap \Xi^{[r]}})$  to bounds on  $\deg(\overline{(Z \cap \mathcal{S} \mathcal{S}^{[r]})^\dagger})$  we show that each component of  $\overline{(Z \cap \mathcal{S} \mathcal{S}^{[r]})^\dagger}$  is also a component of  $\overline{Z \cap \Xi^{[r]}}$ .

While there are several additional steps, the key is to show that if  $k \geq 2$  and  $Z \subseteq \mathbb{A}^k$  is a hypersurface, then  $Z \cap \mathcal{S} \mathcal{S}^{[1,k]}$  is Zariski dense in  $Z$ .

Amusingly, it follows from the fact that the Hecke orbit of any point in  $\mathbb{A}^1$  is infinite that this result is true for  $Z$  weakly special. So, we focus on the case that  $\langle Z \rangle_{ws} = \mathbb{A}^k$ .

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## Density from the Frobenius

- From the reduction to the case that  $Z$  is a non-weakly special hypersurface, we may find an absolutely irreducible polynomial  $G(x_1, \dots, x_k)$  over some finitely generated subring  $R$  of  $K$  with  $I(Z) = (G)$ .
- Find a prime number  $p$  and embeddings  $R \hookrightarrow \mathbb{Z}_p \hookrightarrow K$  so that  $p > \deg(G)$  and  $\mathfrak{Z} = \text{Spec}(\mathbb{Z}_p[x_1, \dots, x_k]/(G))$  is a model of  $Z$ .
- By Hrushovski's theorem on the limit theory of the Frobenius (or Varshavsky's fixed  $p$  version), the set

$$\Sigma := \{(a_1, \dots, a_k) \in \mathbb{A}^k(\mathbb{F}_p^{\text{alg}}) : G(a_1, \dots, a_k) = 0$$

$$\& a_j = a_1^{q^{j-1}} \text{ for } 2 \leq j \leq k \text{ for some power } q \text{ of } p\}$$

is Zariski dense in the special fiber of  $\mathfrak{Z}$ .

- Since  $\Phi_p(x, y) \equiv (y - x^p)(x - y^p) \pmod{p}$ , a Hensel's Lemma argument shows that enough points of  $\Sigma$  lift to a Zariski dense (in  $Z$ ) subset of  $Z \cap \mathcal{S} \mathcal{S}^{[1]}$ .



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# Subtleties

- If  $Z \subseteq \mathbb{A}^k$  is a hypersurface which is not defined over the constants, then the density of  $(Z \cap \mathcal{S}\mathcal{S}^{[1]})^\dagger$  in  $Z$  implies that  $Z \cap \Xi^{[1]}$  is Zariski dense in  $Z$ .
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