ARTIN-SCHREIER EXTENSIONS IN DEPENDENT AND SIMPLE FIELDS

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ABSTRACT. We show that dependent fields have no Artin-Schreier extension, and that simple fields have only a finite number of them.

1. INTRODUCTION

In [20], Macintyre showed that an infinite ω-stable commutative field is algebraically closed; this was subsequently generalized by Cherlin and Shelah to the superstable case; they also showed that commutativity need not be assumed but follows [7]. It is known [28] that separably closed infinite fields are stable; the converse has been conjectured early on [3], but little progress has been made. In 1999 the second author published on his web page a note proving that an infinite stable field of characteristic p at least has no Artin-Schreier extensions, and hence no finite Galois extension of degree divisible by p. This was later generalized to fields without the independence property (dependent fields) by (mainly) the first author.

In the simple case, the situation is even less satisfactory. It is known that an infinite perfect, bounded (i.e. with only finitely many extensions of each degree) PAC (pseudo-algebraically closed: every absolutely irreducible variety has a rational point) field is supersimple of SU-rank one [13]. Conversely, Pillay and Poizat have shown that supersimple fields are perfect and bounded; it is conjectured that they are PAC, but the existence of rational points has only been shown for curves of genus zero (and more generally Brauer-Severi varieties) [23], certain elliptic or hyperelliptic curves [21], and abelian varieties over pro-cyclic fields [22, 16]. Bounded PAC fields are simple [4] and again the converse is conjectured, with even less of an idea on how to prove this [27, Conjecture 5.6.15]; note though that simple and PAC together imply boundedness [5]. In 2006 the third author adapted Scanlon's argument to the simple case and showed that simple fields have only finitely many Artin-Schreier extensions.

In this paper we present the proofs for the simple and the dependent case, and moreover give a criterion for a valued field to be dependent due to the first author.

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2. PRELIMINARIES

Notation 2.1. (1) If k is a field we denote by k^{alg} and k^{sep} its algebraic and separable closures, respectively.

(2) When we write $\bar{a} \in k$ for a tuple $\bar{a} = (a_0, \ldots, a_n)$ we mean that $a_i \in k$ for $i \leq n$.

Definition 2.2. Let K be a field of characteristic $p > 0$. A field extension L/K is called an *Artin-Schreier* extension if $L = K(\alpha)$ for some $\alpha \in L \setminus K$ such that $\alpha^p - \alpha \in K$.

Note that if α is a root of the polynomial $x^p - x - a$ then $\{\alpha, \alpha+1, \ldots, \alpha+p-1\}$ are all the roots of the polynomial. Hence, if $\alpha \notin K$ then L/K is Galois and cyclic of degree p. The converse is also true: if L/K is Galois and cyclic of degree p then it is an Artin-Schreier extension [19, Theorem VI.6.4].

Let K be a field of characteristic $p > 0$, and $\wp : K \to K$ the additive homomorphism given by $\wp(x) = x^p - x$. Then the Artin-Schreier extensions of K are bounded by the number of cosets in $K/\wp(K)$. Indeed, if $K(\alpha)$ and $K(\beta)$ are two Artin-Schreier extensions, then $a = \wp(\alpha)$ and $b = \wp(\beta)$ are both in $K \setminus \wp(K)$, and

$$a - b = \wp(\alpha - \beta) \in \wp(K)$$

implies $\alpha - \beta \in K$ (since K contains $\ker \wp = \mathbb{F}_p$) and hence $K(\alpha) = K(\beta)$.

Remark 2.3. *In fact, the Artin-Schreier extensions of a field k are in bijection with the orbits under the action of $\mathbb{F}_p^\times$ on $k/\wp(k)$.*

Proof. Let $G = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. From [24, X.3] we know that $k/\wp(k)$ is isomorphic to $Hom(G, \mathbb{Z}/p\mathbb{Z})$, and that the isomorphism is induced by taking $c \in k$ to $\varphi_c : G \to \mathbb{Z}/p\mathbb{Z}$, where $\varphi_c(g) = g(x) - x$ for any x satisfying $\wp(x) = c$. Now, every Artin-Schreier extension corresponds to the kernel of a non-trivial element in $Hom(G, \mathbb{Z}/p\mathbb{Z})$. From this it is easy to conclude: Take an Artin-Schreier extension L/k to some φ_c such that $\ker(\varphi_c) = Gal(k^{\mathrm{sep}}, L)$, and from there to the orbit of $c + \wp(k)$. One can check that this is well defined and a bijection. □

We now turn to vector groups.

Definition 2.4. A *vector group* is a group isomorphic to a finite Cartesian power of the additive group of a field.

Fact 2.5. [15, 20.4, Corollary] *A closed connected subgroup of a vector group is a vector group.*

Using infinite Galois cohomology (namely, that $H^1\left(\mathrm{Gal}(k^{\mathrm{sep}}/k), (k^{\mathrm{sep}})^\times\right) = 1$ for a field k, for more on that see [24, X]), one can deduce the following fact:

Corollary 2.6. *Let k be a perfect field, and G a closed connected 1-dimensional algebraic subgroup of $\left(k^{\mathrm{alg}}, +\right)^n$ defined over k, for some $n < \omega$. Then G is isomorphic over k to $\left(k^{\mathrm{alg}}, +\right)$.*

This fact can also proved by combining Théorème 6.6 and Corollaire 6.9 in [9, IV.3.6].

We shall be working with the following group:

Definition 2.7. Let K be a field and $(a_1, \ldots, a_n) = \bar{a} \in K$. Put

$$G_{\bar{a}} = \{(t, x_1, \ldots, x_n) \in K^{n+1} \mid t = a_i\left(x_i^p - x_i\right) \text{ for } 1 \leq i \leq n\}.$$

This is an algebraic subgroup of $(K,+)^{n+1}$.

Recall that for an algebraic group G we denote by G^0 the connected component (subgroup) of the unit element of G.

Lemma 2.8. *Let K be an algebraically closed field. If $\bar{a} \in K$ is algebraically independent, then $G_{\bar{a}}$ is connected.*

Proof. By induction on $n := \text{length}(\bar{a})$. If $n = 1$, then $G_{\bar{a}} = \{(t,x) \mid t = a\cdot(x^p - x)\}$ is the graph of a morphism, hence isomorphic to $\mathbb{A}^1$ and thus connected. Assume the claim for n, and for some algebraically independent $\bar{a} \in K$ of length $n+1$ let $\bar{a}' = \bar{a} \restriction n$. Consider the projection $\pi : G_{\bar{a}} \to G_{\bar{a}'}$. Since K is algebraically closed, π is surjective. Let $H = G^0_{\bar{a}}$ be the identity connected component of $G_{\bar{a}}$. As $[G_{\bar{a}'} : \pi(H)] \le [G_{\bar{a}} : H] < \infty$, it follows that $\pi(H) = G_{\bar{a}'}$ by the induction hypothesis. Assume that $H \neq G_{\bar{a}}$.

Claim. *For every $(t,\bar{x}) \in G_{\bar{a}'}$ there is exactly one x_{n+1} such that $(t,\bar{x},x_{n+1}) \in H$.*

Proof. Suppose for some $(t,\bar{x})$ there were $x^1_{n+1} \neq x^2_{n+1}$ such that $(t,\bar{x},x^i_{n+1}) \in H$ for $i = 1,2$. Hence their difference $(0,\bar{0},\alpha) \in H$. But $0 \neq \alpha \in \mathbb{F}_p$ by definition of $G_{\bar{a}}$. Hence, $(0,\bar{0},1) \in H$, and $(0,\bar{0},\beta) \in H$ for all $\beta \in \mathbb{F}_p$. We know that for every $(t,\bar{x},x_{n+1}) \in G_{\bar{a}}$ there is some x'_{n+1} such that $(t,\bar{x},x'_{n+1}) \in H$; as $x_{n+1} - x'_{n+1} \in \mathbb{F}_p$ we get $(t,\bar{x},x_{n+1}) \in H$ and $G_{\bar{a}} = H$, a contradiction. □

So H is a graph of a function $f : G_{\bar{a}'} \to K$ defined over $\bar{a}$. Now put $t = 1$ and choose $x_i \in K$ for $i \le n$ such that $a_i \cdot (x_i^p - x_i) = 1$. Let $L = \mathbb{F}_p(x_1,\dots,x_n)$ and note that $a_i \in L$ for $i \le n$. Then

$$x_{n+1} := f(1,\bar{x}) \in dcl(a_{n+1},x_1,\dots,x_n) = L(a_{n+1})_{ins}\ ,$$

where $L(a_{n+1})_{ins}$ is the inseparable closure $\bigcup_{n<\omega} L(a_{n+1})^{p^{-n}}$ of $L(a_{n+1})$. Since x_{n+1} is separable over $L(a_{n+1})$, it follows that $x_{n+1} \in L(a_{n+1})$. By assumption, a_{n+1} is transcendental over $\bar{a}'$, whence over L, and so $x_{n+1} \notin L$. Hence $x_{n+1} = h(a_{n+1})/g(a_{n+1})$ for some mutually prime polynomials $g,h \in L[X]$. But then

$$a_{n+1} \cdot \left[h(a_{n+1})^p/g(a_{n+1})^p - h(a_{n+1})/g(a_{n+1})\right] = 1$$

implies

$$a_{n+1} \cdot \left[h(a_{n+1})^p - h(a_{n+1})g(a_{n+1})^{p-1}\right] = g(a_{n+1})^p.$$

This implies that h divides g^p, whence $h \in L$ is constant. Similarly, $g(X)$ divides X, which easily yields a contradiction. □

Corollary 2.9. *If K is perfect and $\bar{a} \in K$ is algebraically independent, then $G_{\bar{a}}$ is isomorphic over K to $(k^{\text{alg}},+)$. In particular, $G_{\bar{a}}(K)$ is isomorphic to $(K,+)$.*

Proof. Over k^{alg} the projection to the first coordinate of $G_{\bar{a}}$ is onto and has finite fibers, so $\dim G_{\bar{a}} = 1$ (as a variety). But then $G^0_{\bar{a}}(k^{\text{alg}})$ is isomorphic over K to $(k^{\text{alg}},+)$ by Corollary 2.6; this isomorphism sends $G^0_{\bar{a}}(K)$ onto $(K,+)$. Finally, $G_{\bar{a}} = G^0_{\bar{a}}$ by Lemma 2.8. □

3. Simple fields

For background on simplicity theory, the interested reader may consult [27]. The only property we shall need is a type-definable variant of Schlichting's Theorem.

Fact 3.1. [27, Theorem 4.5.13] *Let G and Γ be type-definable groups with a definable action of Γ on G, and let $\mathfrak{F}$ be a type-definable Γ-invariant family of subgroups of G. Then there is a Γ-invariant type-definable subgroup $N \leq G$ containing some bounded intersection of groups in $\mathfrak{F}$ such that $[N : N \cap F]$ is bounded for all $F \in \mathfrak{F}$.*

Theorem 3.2. *Let K be a type-definable field in a simple theory. Then K has only boundedly many Artin-Schreier extensions.*

This means that in any elementary extension $\mathfrak{M}$, the number of Artin-Schreier extensions of $K^{\mathfrak{M}}$ remains bounded. In particular, by compactness, if K is definable, it has only finitely many Artin-Schreier extensions.

Proof. If K is finite, then it has precisely one Artin-Schreier extensions. So we may assume it is infinite, and that the model is sufficiently saturated. Let $k = K^{p^\infty} = \bigcap K^{p^n}$, a perfect infinite type-definable sub-field. Let $\wp : K \to K$ be the additive homomorphism given by $\wp(x) = x^p - x$. We shall show that $\wp(K)$ has bounded index in K.

Let $\mathfrak{F} = \{a\wp(K) \mid a \in k\}$; this is a type-definable $k^\times$-invariant family of additive subgroups of K. By Fact 3.1 there exists a type-definable additive $k^\times$-invariant subgroup $N \leq K$ containing a bounded intersection of groups in $\mathfrak{F}$, such that $[N : N \cap F]$ is bounded for all $F \in \mathfrak{F}$.

If N contains $\bigcap_{a\in A} a\wp(K)$ for some bounded $A \subset k$, then for any finite $\bar{a} \in A$ the group $G^0_{\bar{a}}(k)$ is isomorphic to $(k, +)$ by Corollary 2.9. Since k is infinite, the projection to the first coordinate is infinite, as is $\bigcap_{a\in\bar{a}} a\wp(k)$, and even $\bigcap_{a\in A} a\wp(k)$ by compactness, so $N \cap k$ is infinite as well. But $N \cap k$ is $k^\times$-invariant, hence an ideal in k, and must equal k. Since $[N : \wp(K)]$ is bounded, so is $[k : k \cap \wp(K)]$.

Now $a = a^p + \wp(-a)$ for any $a \in K$, whence $K = K^p + \wp(K)$. Assume $K = K^{p^n} + \wp(K)$. Then $K^p = K^{p^{n+1}} + \wp(K^p)$, whence

$$K = K^p + \wp(K) = K^{p^{n+1}} + \wp(K^p) + \wp(K) = K^{p^{n+1}} + \wp(K);$$

by compactness $K = k + \wp(K)$. Thus $[K : \wp(K)] = [k : k \cap \wp(K)]$ is bounded. $\square$

Remark 3.3. *The important category of objects in simple theories are the hyper-definable ones: Quotients of a type-definable set by a type-definable equivalence relation. However, a hyper-definable field is easily seen to be type-definable: If K is given by a partial type π modulo a type-definable equivalence relation E, then for $a, b \in K$ the inequivalence $\neg aEb$ is given by the partial type $\exists x\,[\pi(x) \wedge (a-b)xE1]$. By compactness, E is definable on π.*

4. Dependent fields

Definition 4.1. A theory T has the *independence property* if there is a formula $\varphi(\bar{x}, \bar{y})$ and some model $\mathfrak{M}$ containing tuples $(\bar{a}_i : i \in \omega)$ and $(\bar{b}_I : I \subset \omega)$ such that $\mathfrak{M} \models \varphi(\bar{a}_i, \bar{b}_I)$ if and only if $i \in I$.

Definition 4.2. A theory T is *dependent* if it does not have the independence property.

Remark 4.3. *Let k be a field, and let $f : k \to k$ be an additive polynomial, i.e. $f(x+y) = f(x) + f(y)$. Then f is of the form $\sum a_i x^{p^i}$. Furthermore, if k is algebraically closed and $\ker(f) = \mathbb{F}_p$, then $f = a \cdot (x^p - x)^{p^n}$ for some $n < \omega$ and $a \in k$.*

Proof. The first part appears in [11, Proposition 1.1.5]. Assume now that k is algebraically closed and $|\ker(f)| = p$. If $a_0 \neq 0$, then $(f, f') = 1$, hence f has no multiple factors and $deg(f) = p$. If $a_0 = 0$, then $f = (g(x))^p$ for some additive polynomial g with $|\ker(g)| = p$. So by induction $f = (a_0 x + a_1 x^p)^{p^n}$ for some $n < \omega$. If moreover $\ker(f) = \mathbb{F}_p$, then $a_0 + a_1 = 0$ hence $f = a \cdot (x^p - x)^{p^n}$ for some $a \in k$. □

Theorem 4.4. *Let K be an infinite dependent field. Then K is Artin-Schreier closed.*

Proof. We may assume that K is $\aleph_0$-saturated, and we put $k = K^{p^\infty}$, a type-definable infinite perfect sub-field. For $a \in k$ let

$$H_a = \{t \in K \mid \exists x \in K \ a \cdot (x^p - x) = t\}.$$

By dependency the Baldwin-Saxl condition [1] holds, which means that there is $n < \omega$ such that for every $(n+1)$-tuple $\bar{a}$, there is a sub-n-tuple $\bar{a}'$ with $\bigcap_{a \in \bar{a}} H_a = \bigcap_{a \in \bar{a}'} H_{\bar{a}'}$. This implies that the projection $\pi : G_{\bar{a}}(K) \to G_{\bar{a}'}(K)$ is onto, where

$$G_{\bar{a}} = \{(t, x_1, \ldots, x_n) \in K^{n+1} \mid t = a_i\,(x_i^p - x_i) \text{ for } 1 \leq i \leq n\}$$

is the group defined in Definition 2.7. We fix some algebraically independent $(n+1)$-tuple $\bar{a} \in k$.

By Corollary 2.9 we have algebraic isomorphisms $G_{\bar{a}} \to (k^{\mathrm{alg}}, +)$ and $G_{\bar{a}'} \to (k^{\mathrm{alg}}, +)$ over k. Hence we can find an algebraic map ρ over k which makes the following diagram commute:

$$\begin{array}{ccc} G_{\bar{a}}(k^{\mathrm{alg}}) & \xrightarrow{\ \pi\ } & G_{\bar{a}'}(k^{\mathrm{alg}}) \\ \downarrow & & \downarrow \\ (k^{\mathrm{alg}}, +) & \xrightarrow{\ \rho\ } & (k^{\mathrm{alg}}, +) \end{array}$$

As all groups and maps are defined over $k \subseteq K$, we can restrict to K. But $\pi \restriction G_{\bar{a}}(K)$ is onto $G_{\bar{a}'}(K)$, so $\rho \restriction K$ must be onto as well. Moreover,

$$|\ker(\rho)| = |\ker(\pi)| = |(0, \bar{0}) \times \mathbb{F}_p| = p\,;$$

since $\ker(\pi)$ is contained in $G_{\bar{a}}(K)$, this remains true in the restrictions to K. Finally, ρ is a group homomorphism, i.e. additive, and a polynomial, as it is an algebraic morphism of $(k^{\mathrm{alg}}, +)$.

Suppose that $0 \neq c \in \ker(\rho) \subseteq K$, and put $\rho'(x) = c^{-1} \cdot \rho(x)$. Then ρ' is an additive polynomial whose kernel is $\mathbb{F}_p$. By Remark 4.3 there are $a \in k$ and $n < \omega$ such that $\rho'(x) = a \cdot (x^p - x)^{p^n}$. As $\rho' \restriction K$ is onto K, for any $y \in K$ there is some $x \in K$ with

$$a \cdot (x^p - x)^{p^n} = a \cdot y^{p^n},$$

so $\wp(x) = x^p - x = y$ and we are done.

In fact, n must be 0, as the degree of π (as algebraic morphism) is p, and so is the degree of ρ', since the vertical arrows are algebraic isomorphisms. □

Corollary 4.5. *If K is an infinite dependent field of characteristic $p > 0$ and L/K is a finite separable extension, then p does not divide $[L : K]$.*

Proof. Assume not, and let L' be the normal closure of L/K. Then $p \mid [L' : K]$, so we may assume that L/K is Galois. Let $G \leq \mathrm{Gal}(L/K)$ be a subgroup of order p, and let $K^G \subseteq L$ be its fixed field. As K^G is interpretable in K, it is also dependent. But L/K^G is an Artin-Schreier extension, contradicting Theorem 4.4. □

Corollary 4.6. *Let K be an infinite dependent field of characteristic $p > 0$. Then K contains $\mathbb{F}_p^{\mathrm{alg}}$.*

Proof. Let $k = K \cap \mathbb{F}_p^{\mathrm{alg}}$, the relative algebraic closure of $\mathbb{F}_p$ in K. As K is Artin-Schreier closed, so is k. Hence k is infinite, perfect, and pseudo-algebraically closed. But [10, Theorem 6.4] of Duret states that a field with a relatively algebraically closed PAC subfield which is not separably closed has the independence property. Hence k is algebraically closed, i.e. $k = \mathbb{F}_p^{\mathrm{alg}}$. □

One might wonder what happens for a type-definable field in a dependent theory. We were unable to generalize our theorem to this case. However, one easily sees:

Proposition 4.7. *Let K be a type-definable field in a dependent theory. Then K has either no, or unboundedly many Artin-Schreier extensions.*

Proof. By [25] (another presentation appears in [14, Proposition 6.1]) there is a minimal type-definable subgroup K^{00} of $(K, +)$ of bounded index. As for any $\lambda \in K^\times$, the multiplicative translate λG^{00} is also a type-definable additive subgroup of bounded index, K^{00} is an ideal of bounded index and must therefore be equal to K. On the other hand, the image of $\wp$ is a type-definable subgroup of $(K, +)$. Remark 2.3 tells us that it has bounded index if and only if there are boundedly many Artin-Schreier extensions. But if it has bounded index, then it contains $K^{00} = K$, and K is Artin-Schreier closed. □

In an attempt to prove the theorem for type-definable fields, we found the following lemma concerning type-definable groups in dependent theories:

Definition 4.8. Let G be a group, $\mathfrak{H}$ a family of subgroups of G and κ a cardinal. The κ-almost intersection is the subgroup

$$\bigcap^{\kappa} \mathfrak{H} = \{g \in G \mid \mathrm{card}(\{H \in \mathfrak{H} \mid g \notin H\}) < \kappa.\}.$$

Proposition 4.9. *Let G be a type-definable group in a dependent theory. Then for any type-definable family $\mathfrak{H}_0$ of subgroups of G there is a cardinal κ_0 such that for any regular cardinal $\kappa \geq \kappa_0$, and subfamily $\mathfrak{H} \subseteq \mathfrak{H}_0$ in any elementary extension, the intersection $\bigcap \mathfrak{H}$ is a subintersection of size less than κ intersected with the κ-almost intersection $\bigcap^{\kappa} \mathfrak{H}$. In fact, if κ_1 is a bound for the number of parameters defining a group in $\mathfrak{H}_0$ and every $g \in G$ is a tuple of length κ_2, then we can take $\kappa_0 = |T|^+ + \kappa_1 + \kappa_2$.*

Proof. Let $\kappa \geq \kappa_0$ be regular. Sets of cardinality less than κ will be called *small*. Assume that there is some family $\mathfrak{H} = \{H_i \mid i < \lambda\}$ of uniformly type-definable subgroups of G which is not equal to a small subintersection intersected with the κ-almost intersection. For $g \in G$ define $J_g = \{i < \lambda \mid g \in H_i\}$. So $g \in \bigcap^\kappa \mathfrak{H}$ if and only if $\lambda \setminus J_g$ is small.

We shall define inductively on $i < \kappa$ elements $g_i \in \bigcap^\kappa \mathfrak{H}$, subsets $I_i \subseteq \lambda$ and ordinals $\alpha_i < \lambda$, such that

(1) $I_i \cap [0, \alpha_i] = \emptyset$,
(2) I_i is decreasing,
(3) α_i is increasing,
(4) $I_i \subseteq J_{g_i}$,
(5) $\bigcap_{j \in \lambda \setminus I_i} H_j \supseteq \bigcap^\kappa \mathfrak{H} \cap \bigcap_{j \in J} H_j$ for some small $J \subseteq \lambda$, and
(6) for $i \neq j$ we have $g_i \in H_{\alpha_j} \setminus H_{\alpha_i}$.

Assume that g_j, I_j, α_j have been chosen for $j < i$. Put $I'_i = \bigcap_{j<i} I_j$ (where $I'_0 = \lambda$). Let $\alpha_i \in I'_i$ be minimal such that there is some element

$$g_i \in \Big(\bigcap^\kappa \mathfrak{H} \cap \bigcap_{j<i} H_{\alpha_j}\Big) \setminus H_{\alpha_i}.$$

Such an α_i must exist, as otherwise $\bigcap^\kappa \mathfrak{H} \cap \bigcap_{j<i} H_{\alpha_j} \subseteq \bigcap_{j \in I'_i} H_j$, so

$$\bigcap \mathfrak{H} = \bigcap^\kappa \mathfrak{H} \cap \bigcap_{j<i} H_{\alpha_j} \cap \bigcap_{j \in \lambda \setminus I'_i} H_j.$$

But now $[0, i)$ is small, and by (5) and regularity of κ there is a small J with

$$\bigcap_{j \in \lambda \setminus I'_i} H_j \supseteq \bigcap^\kappa \mathfrak{H} \cap \bigcap_{j \in J} H_j.$$

This contradicts our assumption on $\mathfrak{H}$.

Let $I_i = \{j \in I'_i \mid j > \alpha_i\} \cap J_{g_i}$. This takes care of (1) and (4). Now (2) is obvious, and (3) follows from (1) in the induction. By the minimality of α_i,

$$\bigcap^\kappa \mathfrak{H} \cap \bigcap_{j<i} H_{\alpha_j} \subseteq \bigcap_{j \in I'_i \cap [0, \alpha_i)} H_j.$$

So

$$\bigcap_{j \in [0,\alpha_i]} H_j \cap \bigcap_{j \in \lambda \setminus I'_i} H_j = \bigcap_{j \in (\lambda \setminus I'_i) \cup ([0,\alpha_i] \cap I'_i)} H_j \supseteq \bigcap^\kappa \mathfrak{H} \cap \bigcap_{j \leq i} H_{\alpha_j} \cap \bigcap_{j \in J} H_j.$$

As

$$\bigcap_{j \in \lambda \setminus I_i} H_j = \bigcap_{j \in \lambda \setminus J_{g_i}} H_j \cap \bigcap_{j \in [0,\alpha_i]} H_j \cap \bigcap_{j \in \lambda \setminus I'_i} H_j$$

we get (5). Finally for $j < i$ we have $g_i \in H_{\alpha_j} \setminus H_{\alpha_i}$ by choice of g_i, and for $j > i$ we have $g_i \in H_{\alpha_j}$ since $\alpha_j \in I_i \subset J_{g_i}$, so (6) holds as well.

Now the usual argument works: Since $d_1 \cdot g_i \cdot d_2 \notin H_{\alpha_i}$ for any $d_1, d_2 \in H_{\alpha_i}$, by compactness there is some formula $\varphi_i(x, b_i)$ containing H_{α_i} such that $\neg\varphi_i(d_1 g_i d_2, b_i)$ for all $d_1, d_2 \in H_{\alpha_i}$. As $\kappa > |T|$, we can extract an infinite subset I of κ such that the same formula $\varphi(x, y)$ will work for all $i \in I$. Now for any finite subset s of I let g_s be the product $\prod_{i \in s} g_i$. Then $\varphi(g_s, b_i)$ if and only if $i \notin s$, contradicting dependency. $\square$

We can, however, prove that a type-definable field is Artin-Schreier closed under a stronger hypothesis.

Definition 4.10. Call a type-definable group G *strongly connected* when $G^{00} = G$.

Note that if $\pi : G \to H$ is a definable surjective group homomorphism and G is strongly connected, then so is H, since $|G : \pi^{-1}(H^{00})|$ is bounded by $|H : H^{00}|$ and π is onto.

Theorem 4.11. *Let K be a type-definable field in a dependent theory such that there is no infinite decreasing sequence of type-definable additive subgroups, each of unbounded index in its predecessor. Then K is Artin-Schreier closed.*

Proof. We work in a saturated model. Let $\bar{a} = (a_i : i < \omega)$ be a sequence of algebraically independent elements from $k = \bigcap K^{p^n}$. Let $H_i = a_i \cdot \wp(K)$, and recall that $\bigcap_{j<n} H_{i_j} = \pi_1(G_{(a_{i_0},\dots,a_{i_{n-1}})}(K))$ for all $i_0 < \dots < i_{n-1}$, where π_1 is the projection to the first coordinate. Since $G_{(a_{i_0},\dots,a_{i_{n-1}})}(K)$ is isomorphic (over k) to $(K,+)$ and we mentioned in 4.7 that the latter is strongly connected, $\bigcap_{j<n} H_{i_j}$ is strongly connected, too. By assumption, there is some n such that $\bigcap_{i<n} H_i = \bigcap_{i<n+1} H_i$. Now proceed as in the proof of Theorem 4.4. □

Remark 4.12. *Saharon Shelah has shown* [26] *that this condition holds when T is strongly*[2] *dependent.*

5. Some results on dependent valued fields

Here we find a nice characterization of "nice" dependent valued fields of characteristic $p > 0$. First we recall the definitions and notations:

Definition 5.1. A valued field is a pair (K, v) where K is a field and $v : K \to \Gamma \cup \{\infty\}$ for an ordered group Γ such that:

(1) $v(x) = \infty$ if and only if $x = 0$,
(2) $v(x \cdot y) = v(x) + v(y)$, and
(3) $v(x + y) \geq \min\{v(x), v(y)\}$.

If (K, v) is a valued field, then $\Gamma = v(K^\times)$ is the *valuation group*, $\mathcal{O}_K = \{x \in K \mid v(x) \geq 0\}$ is the (local) ring of *integers*, $\mathfrak{m}_K = \{x \in K \mid v(x) > 0\}$ is its *maximal ideal*, and $k = \mathcal{O}_K/\mathfrak{m}_K$ is the *residue field*. As a structure we think of it as a 3-sorted structure (K, Γ, k) equipped with the valuation map $v : K^\times \to \Gamma$, and the quotient map $\pi : \mathcal{O}_K \to k$. Other interpretations are known to be equivalent (i.e. bi-interpretable, and hence to preserve properties such as dependency).

In [8] Delon gave the following characterization of Henselian dependent valued fields of characteristic 0.

Fact 5.2. [8] *Let (K, v) be a Henselian valued field of characteristic 0. Then (K, v) is dependent if and only if the residue field k is dependent.*

Historically, this theorem stated that the valuation group must also be dependent, but by a result of Gurevich and Schmitt [12], every ordered abelian group is dependent.

Here we discuss valued fields of characteristic p, i.e. with $\mathrm{char}(K) = \mathrm{char}(k) = p$.

Proposition 5.3. *If (K, v) is a dependent valued field of characteristic $p > 0$, then the residue field contains $\mathbb{F}_p^{\text{alg}}$.*

Proof. Suppose $\pi(a) \in k$ with $a \in \mathcal{O}_K$. Since K is Artin-Schreier closed, there is $b \in K$ with $b^p - b = a$. If $v(b) < 0$, then $v(b^p) = p\,v(b) < v(b)$, whence $v(b^p - b) = v(b^p) < 0$, contradicting $v(a) \geq 0$. Hence $v(b) \geq 0$ and $b \in \mathcal{O}_K$. Thus $\pi(b) \in k$, and $\pi(b)^p - \pi(b) = \pi(a)$. In other words, k is also Artin-Schreier closed, and hence infinite; since it is interpretable, it is dependent, and contains $\mathbb{F}_p^{\text{alg}}$ by Corollary 4.6. □

Proposition 5.4. *If (K, v) is a dependent valued field of characteristic $p > 0$, then the valuation group Γ is p-divisible.*

Proof. Let $0 > \alpha \in \Gamma$. So $\alpha = v(a)$ for some $a \in K^\times$. As K is Artin-Schreier closed, there is some $b \in K^\times$ such that $b^p - b = a$. Clearly $v(b) \geq 0$ is impossible. Hence $v(b^p) = p\; v(b) < v(b)$, and

$$\alpha = v(a) = v(b^p - b) = \min\{v(b^p), v(b)\} = v(b^p) = p\,v(b).$$

So α is p-divisible, as is Γ (for α positive, consider $-\alpha$). □

As a corollary we obtain a result of Cherlin [6].

Corollary 5.5. *$\mathbb{F}_p((t))$ is independent, and so is $\mathbb{F}_p^{\text{alg}}((t))$.*

Propositions 5.3 and 5.4 are also sufficient for a valued field to be dependent, under certain conditions. In order to explain these conditions, we give two definitions.

Definition 5.6. A valued field (K, v) of characteristic $p > 0$ is called a Kaplansky field if it satisfies:

(1) The valuation group Γ is p-divisible,
(2) The residue field k is perfect, and does not admit a finite separable extension divisible by p.

This definition is taken from the unpublished book on valuation theory by Franz-Viktor Kuhlmann [18, 13.11]. It is first-order expressible, as the second condition is equivalent to saying that for every additive polynomial $f \in k[x]$, and every $a \in k$, there is a solution to $f(x) = a$ in k (for a proof, see [17, Theorem 5]).

Definition 5.7. A valued field (K, v) is called algebraically maximal if it does not admit any non-trivial algebraic immediate extension (i.e. keeping both the residue field and the valuation group).

This is also first order axiomatizable [18, Chapter 14, Section 2]. It always implies Henselianity, and is equivalent to it in characteristic 0. In characteristic p, it is weaker than being Henselian and defectless ([18, 9.39]).

We shall use the following result of Bélair.

Fact 5.8. [2, Corollaire 7.6] *A valued field K of characteristic p which is Kaplansky and algebraically maximal is dependent if and only if k is dependent.*

Finally, we have:

Theorem 5.9. *Let (K, v) be an algebraically maximal valued field of characteristic p whose residue field k is perfect. Then (K, v) is dependent if and only if k is dependent and infinite and Γ is p-divisible.*

Proof. If (K, v) is dependent then k is infinite (it even contains $\mathbb{F}_p^{\mathrm{alg}}$), and dependent, and Γ is p-divisible, by Propositions 5.3 and 5.4. On the other hand, if k is dependent and infinite, by Corollary 4.5 we get that (K, v) is Kaplansky and we can apply fact 5.8. □

It is interesting to note the connection to Kuhlmann's notion of a tame valued field (see [18, Chapter 13, Section 9]). A valued field (K, v) is called tame if and only if it is algebraically maximal, Γ is p-divisible and k is perfect. Note the difference between this and Kaplansky.

We get as an immediate corollary:

Corollary 5.10. *Let (K, v) be an algebraically maximal dependent valued field. Then K is tame if and only if K is Kaplansky, if and only if k is perfect.*

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