INFINITE STABLE FIELDS ARE ARTIN-SCHREIER CLOSED

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ABSTRACT. We note that if K is an infinite stable field of characteristic $p > 0$, then the Artin-Schreier map $\wp : K \to K$ given by $x \mapsto x^p - x$ is surjective. Consequently, K has no finite Galois extensions of degree divisible by p.

An old theorem of Macintyre asserts that an ω-stable infinite field is algebraically closed [4]. Cherlin and Shelah generalized this theorem showing that an infinite superstable division algebra must be a commutative algebraically closed field [2]. As separably closed fields are stable [5], one might ask whether an infinite stable field must be separably closed. We note that at the very least if K is stable and infinite of characteristic p, then K has no finite Galois extensions of degree divisible by p.

I thank Anand Pillay for calling to my attention an error in an earlier version of this note.

Proposition 1. *Let K be an infinite stable field of characteristic $p > 0$. Then the Artin-Schreier map $\wp : K \to K$ given by $x \mapsto x^p - x$ is surjective.*

Proof: As the hypotheses and conclusion are properties of the theory of K, we may assume that K is $\aleph_1$-saturated. Let $k := K^{p^\infty} := \bigcap_{n=1}^\infty K^{p^n}$. Let $\varphi(x, y) := (\exists z) y(z^p - z) = x$. Then, for any $a \in K$, $\varphi(K, a) = a\wp(K)$. That is, $\{a\wp(K) | a \in K^\times\}$ is family of uniformly definable subgroups of $(K, +)$. As K is stable, the group $I := \bigcap_{a \in k^\times} a\wp(K)$ is equal to a finite subintersection of the form $\bigcap_{j=1}^n a_j \wp(K)$ for some $a_1, \ldots, a_n \in k^\times$. Let G be the algebraic subgroup of the $(n+1)^{\text{st}}$ power of the additive group defined by $\bigwedge_{j=1}^n t = a_j(X_j^p - X_j)$. Note that the dimension of G is at least one. As k is perfect, the connected component of G is isomorphic to the additive group over k. In particular, $G(k)$ is infinite. So, we can find some nonzero $t \in \bigcap_{j=1}^n a_j \wp(k)$. However, $I \cap k$ is an ideal of the field k; so $I \supseteq k$. On the other hand, $\wp(k) \geq I \cap k$. Thus, $\wp(k) = k$.

By compactness, there is some n so that $\wp(K) \supseteq K^{p^n}$. Let $a \in K$, then $a^{p^n} \in \wp(K)$. Let $\alpha \in K$ so that $\alpha^{p^n} - \alpha = x^{p^n}$. Then, $\wp(\alpha^{p^{-n}}) = a$. Note that the extension $K(\alpha^{p^{-n}})/K$ is purely inseparable. However, the equation $X^p - X - a = 0$ is separable, so the extension $K(\alpha^{p^{-n}})/K$ is also separable. Therefore, $\alpha^{p^{-n}} \in K$ so that $K = \wp(K)$. ✠

My original proof (which I have forgotten) of the main proposition exploited an algebraic lemma of Cherlin [1]. His calculation is hidden in the general fact on rationality of certain linear groups quoted in the above proof.

Corollary 2. *If K is an infinite stable field of characteristic $p > 0$ and L/K is a finite separable extension, then p does not divide $[L : K]$.*

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Proof: Let M/L be the Galois closure of L over K. If p divided $[L:K]$, then it also divides $[M:K] = \#\mathrm{Gal}(M/K)$. Let $G \leq \mathrm{Gal}(M/K)$ be a subgroup of order p and $K' := M^G$, the fixed field of G. The extension M/K' is Galois of degree $p = char K'$; so it is given by adjoining an Artin-Schreier root. In particular, $\wp(K') \neq \wp(K')$. However, stability is preserved by interpretations and K' is interpretable in K. This contradicts the proposition. ✠

Corollary 3. *If K is an infinite stable field of characteristic $p > 0$, then K contains $\mathbb{F}_p^{\mathrm{alg}}$.*

Proof: Let $k := K \cap \mathbb{F}_p^{\mathrm{alg}}$. By our proposition, K is closed under taking Artin-Schreier roots. Thus, k contains $\bigcup_{n=0}^{\infty} \mathbb{F}_{p^{p^n}}$ and is therefore infinite. By the Lang-Weil estimates, every infinite subfield of $\mathbb{F}_p^{\mathrm{alg}}$ is pseudo-algebraically closed (PAC). By a theorem of Duret [3], every relatively algebraically closed PAC subfield of a field without the independence property is algebraically closed. Hence, $k = \mathbb{F}_p^{\mathrm{alg}}$. ✠

References

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