General method

If \( R := \{ (x, y) \mid a \leq x \leq b, g(x) \leq y \leq h(x) \} \) then

\[
\int \int_{R} f(x, y) \, dx \, dy = \int_{a}^{b} \left( \int_{g(x)}^{h(x)} f(x, y) \, dy \right) dx
\]

To compute the iterated integral, first find \( F(x, y) \) for which \( \frac{\partial F}{\partial y} = f(x, y) \) so that

\[
\int_{g(x)}^{h(x)} f(x, y) \, dy = F(x, y)|_{y=g(x)}^{y=h(x)} = F(x, h(x)) - F(x, g(x))
\]

Then find \( \Phi(x) \) for which \( \Phi'(x) = F(x, h(x)) - F(x, g(x)) \) and we find

\[
\int_{a}^{b} (F(x, h(x)) - F(x, g(x))) \, dx = \Phi(b) - \Phi(a)
\]

Putting these together, we have

\[
\int \int_{R} f(x, y) \, dx \, dy = \Phi(b) - \Phi(a)
\]
Example

Let $R = \{(x, y) \mid 1 \leq x \leq 2, 1 \leq y \leq e^x\}$. Compute

$$\int \int_{R} \frac{x}{y} \, dx \, dy$$

Solution

$$\int \int_{R} \frac{x}{y} \, dx \, dy = \int_{1}^{2} \left( \int_{1}^{e^x} \frac{x}{y} \, dy \right) \, dx$$

$$= \int_{1}^{2} \left( x \ln(y) \right|_{y=1}^{y=e^x} \, dx$$

$$= \int_{1}^{2} (x \ln(e^x) - x \ln(1)) \, dx$$

$$= \int_{1}^{2} x^2 \, dx$$

$$= \frac{1}{3} x^3 \big|_{x=1}^{x=2}$$

$$= 8 - \frac{1}{3}$$

$$= \frac{7}{3}$$
Another example

Let \( R = \{(x, y) \mid 1 \leq x \leq 3, \ln(x) \leq y \leq 4\} \). Compute

\[
\int \int_R xe^y dx dy
\]

Solution

\[
\int \int_R e^{x+y} dx dy = \int_1^3 \left( \int_{\ln(x)}^4 xe^y dy \right) dx
\]

\[
= \int_1^3 \left( xe^y \right|_{y=\ln(x)}^{y=4} dx
\]

\[
= \int_1^3 (x^2 - e^4 x) dx
\]

\[
= \left( \frac{1}{3} x^3 - \frac{e^4}{2} x^2 \right|_{x=1}^{x=3}
\]

\[
= 9 - \frac{9}{2} e^4 - \frac{1}{3} + e^4 \frac{1}{2}
\]

\[
= \frac{26}{3} - 4e^4
\]
Changing variables
If the region $R$ is defined by a constant bound on $y$ with the bound on $x$ varying as a function of $y$, then one may compute a double integral over $R$ as an iterated integral where one first integrates with respect to $x$ and then with respect to $y$.

Example
Let $R = \{(x, y) \mid y \leq x \leq 2y, 1 \leq y \leq 2\}$. Compute
\[
\int \int_{R} e^{x+y} \, dx \, dy
\]
\[
\int \int_R e^{x+y} \, dx \, dy = \int_1^2 \left( \int_y^{2y} e^{x+y} \, dx \right) \, dy \\
= \int_1^2 \left( \int_y^{2y} e^y e^x \, dx \right) \, dy \\
= \int_1^2 \left( e^y e^x \left|_{x=y}^{x=2y} \right. \right) \, dy \\
= \int_1^2 \left( e^y e^{2y} - e^y e^y \right) \, dy \\
= \int_1^2 \left( e^{3y} - e^{2y} \right) \, dy \\
= \left[ \frac{1}{3} e^{3y} - \frac{1}{2} e^{2y} \right]_{y=1}^{y=3} \\
= \frac{1}{3} e^9 - \frac{1}{2} e^6 - \frac{1}{3} e^3 + \frac{1}{2} e^2
\]

Rectangles and Fubini’s Theorem

If the region \( R \) is a rectangle with horizontal and vertical sides relative to the coordinate axes, the one may compute a double integral over \( R \) as an iterated integral by first integrating with respect to \( x \) and then with respect to \( y \) or vice versa.
Example

Let \( R = \{(x, y) \mid 0 \leq x \leq 2, 2 \leq y \leq 5\} \). Compute

\[
\int \int_R \frac{x^2}{y} \, dxdy
\]

Solution

\[
\int \int_R \frac{x^2}{y} \, dxdy = \int_0^2 \left( \int_2^5 \frac{x^2}{y} \, dy \right) dx
\]

\[
= \int_0^2 \left( x^2 \ln(y) \right|_{y=2}^{y=5} \, dx
\]

\[
= \int_0^2 \left[ \ln(5)x^2 - \ln(2)x^2 \right] \, dx
\]

\[
= \int_0^2 \left[ \ln\left(\frac{5}{2}\right)x^2 \right] \, dx
\]

\[
= \left[ \frac{5}{2} \frac{1}{3} x^3 \right]_0^2
\]

\[
= \frac{8}{3} \ln\left(\frac{5}{2}\right)
\]
Another solution

\[ \int \int_{R} \frac{x^2}{y} \, dx \, dy = \int_{2}^{5} \left( \int_{0}^{2} \frac{x^2}{y} \, dx \right) \, dy \]

\[ = \int_{2}^{5} \frac{1}{3} \frac{x^3}{y} \bigg|_{x=0}^{x=2} \, dy \]

\[ = \int_{2}^{5} \frac{8}{3y} \, dy \]

\[ = \frac{8}{3} \ln(y) \bigg|_{y=2}^{y=5} \]

\[ = \frac{8}{3} (\ln(5) - \ln(2)) \]

\[ = \frac{8}{3} \ln\left(\frac{5}{2}\right) \]