Section 7.7: Double integrals

If $R$ is some region in the plane and $f(x, y)$ is a (continuous) function taking positive values, then the volume of \( \{(x, y, z) \mid (x, y) \text{ in } R \text{ and } 0 \leq z \leq f(x, y)\} \) is the integral

\[
\int \int_R f(x, y) \, dx \, dy
\]

More generally, if $f(x, y)$ is any continuous function, then \( \int \int_R f(x, y) \, dx \, dy \) is the signed volume of the solid bounded by $f$.

Example

If $R := \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ is the unit square and $f(x, y) \equiv 1$, then the solid bounded by $f$ over $R$ is the unit cube. So, \( \int \int_R f(x, y) \, dx \, dy = 1 \).
Iterated integrals

If \( R = \{(x, y) \mid a \leq x \leq b, g(x) \leq y \leq h(x)\} \) (where \( a \leq b \) are constants and \( g \) and \( h \) are continuous functions of \( x \)), then

\[
\int \int_{R} f(x, y) \, dx \, dy = \int_{a}^{b} \left( \int_{g(x)}^{h(x)} f(x, y) \, dy \right) \, dx
\]

Here, \( F(x) = \int_{g(x)}^{h(x)} f(x, y) \, dy \) is itself a function of \( x \).

Example

Suppose \( f(x, y) = xy \) and \( R = \{(x, y) \mid 1 \leq x \leq 2, x \leq y \leq x^2\} \). Compute

\[
\int \int_{R} f(x, y) \, dx \, dy
\]
\[ \int \int_R f(x, y) \, dx \, dy = \int \int_{\{(x,y) \mid 1 \leq x \leq 2, x \leq y \leq x^2\}} xy \, dx \, dy \]

\[ = \int_{1}^{2} \left( \int_{x}^{x^2} xy \, dy \right) \, dx \]

\[ = \int_{1}^{2} \left( \frac{1}{2} xy^2 \mid y = x \right) \, dx \]

\[ = \int_{1}^{2} \left( \frac{1}{2} x(x^2)^2 - \frac{1}{2} x^2(x) \right) \, dx \]

\[ = \int_{1}^{2} \left( \frac{1}{2} x^5 - \frac{1}{2} x^3 \right) \, dx \]

\[ = \left( \frac{1}{12} x^6 - \frac{1}{8} x^4 \right) \bigg|_{x=1}^{x=2} \]

\[ = \left( \frac{64}{12} - \frac{16}{8} - \frac{1}{12} + \frac{1}{8} \right) \]

\[ = \frac{27}{8} \]

Another example

Compute

\[ \int \int_{\{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq e^x\}} e^x \, dx \, dy \]
Solution

\[ \int \int_{\{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq e^x \}} ye^x \, dx \, dy = \int_0^1 (\int_0^{e^x} e^x \, dy) \, dx \]

= \int_0^1 (ye^x \mid_{y=0}^{y=e^x}) \, dx

= \int_0^1 e^{2x} \, dx

= \left( \frac{1}{2} e^{2x} \right) \mid_{x=0}^{x=1}

= \frac{1}{2} (e^2 - 1)

Yet another example

Compute

\[ \int \int_{\{(x,y) \mid 1 \leq x \leq 4, \sqrt{x} \leq y \leq x^3 \}} \frac{x}{y^2} \, dx \, dy \]
Solution

\[ \int \int_{\{(x,y) \mid 1 \leq x \leq 4, \sqrt{x} \leq y \leq x^2 \}} \frac{x^2}{y} \, dx \, dy = \int_1^4 \left( \int_{\sqrt{x}}^{x^2} \frac{x}{y} \, dy \right) \, dx \]
\[ = \int_1^4 \left( \frac{x}{y} \right)_{y=\sqrt{x}}^{y=x^2} \, dx \]
\[ = \int_1^4 \left( \sqrt{x} - \frac{1}{x} \right) \, dx \]
\[ = \left( \frac{2}{3} x^{3/2} + \frac{1}{x} \right)_{x=1}^{x=4} \]
\[ = \left( \frac{16}{3} + \frac{1}{4} - \frac{2}{3} - 1 \right) \]
\[ = \frac{47}{12} \]