12.4: Exponential and normal random variables Exponential density function

Given a positive constant $k > 0$, the exponential density function (with parameter k) is

$$f(x) = \begin{cases} ke^{-kx} \text{ if } x \geq 0 \\ 0 \text{ if } x < 0 \end{cases}$$

Expected value of an exponential random variable

Let X be a continuous random variable with an exponential density function with parameter k.

Integrating by parts with $u = kx$ and $dv = e^{-kx}dx$ so that $du = kdx$ and $v = \frac{-1}{k}e^{-kx}$, we find

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_{0}^{\infty} kxe^{-kx}dx \\ &= \lim_{r \to infty} [-xe^{-kx} - \frac{1}{k}e^{-kx}]|_0^r \\ &= \frac{1}{k} \end{aligned}$$

Variance of exponential random variables

Integrating by parts with $u = kx^2$ and $dv = e^{-kx}dx$ so that $du = 2kxdx$ and $v = \frac{-1}{k}e^{-kx}$, we have

$$\begin{aligned}\int_0^\infty x^2 e^{-kx} dx &= \lim_{r\to\infty} ([-x^2 e^{-kx}]|_0^r + 2\int_0^r xe^{-kx}dx) \\ &= \lim_{r\to\infty} ([-x^2 e^{-kx} - \frac{2}{k}xe^{-kx} - \frac{2}{k^2}e^{-kx}]|_0^r) \\ &= \frac{2}{k^2}\end{aligned}$$

So, $\text{Var}(X) = \frac{2}{k^2} - E(X)^2 = \frac{2}{k^2} - \frac{1}{k^2} = \frac{1}{k^2}$.

Example

Exponential random variables (sometimes) give good models for the time to failure of mechanical devices. For example, we might measure the number of miles traveled by a given car before its transmission ceases to function. Suppose that this distribution is governed by the exponential distribution with mean $100,000$. What is the probability that a car's transmission will fail during its first $50,000$ miles of operation?

Solution

$$\begin{aligned}\Pr(X \leq 50,000) &= \int_0^{50,000} \frac{1}{100,000} e^{\frac{-1}{100,000}x} dx \\ &= -e^{\frac{-x}{100,000}} \Big|_0^{50,000} \\ &= 1 - e^{\frac{-1}{2}} \\ &\approx .3934693403\end{aligned}$$

Normal distributions

The normal density function with mean μ and standard deviation σ is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

As suggested, if X has this density, then $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

The *standard normal density* function is the normal density function with $\mu = \sigma = 1$. That is,

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}x^2}$$

Taylor expansion for the normal cumulative distribution function

Let $f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}x^2}$ be the standard normal density function and let $F(x) = \int_{-\infty}^{x} f(t)dt$ be the standard normal *cumulative distribution* function.

We compute a Taylor series expansion,

$$\begin{aligned} G(x) &= \int \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}x^2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n} x^{2n} dx \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n(2n+1)} x^{2n+1} \\ &= \frac{1}{\sqrt{2\pi}} (x - \frac{1}{6}x^3 + \frac{1}{40}x^5 - \frac{1}{336}x^7 + \cdots) \end{aligned}$$

So $F(x) = G(x) + C$ for some C. As 0 is the expected value, we need $\frac{1}{2} = F(0) = G(0) + C = C$.

Example

If the continuous random variable X is normally distributed, what is the probability that it takes on a value of more than a standard deviations above the mean?

Solution

Via a change of variables, we may suppose that X is normally distributed with respect to the standard normal distribution. Let F be the cumulative distribution function for the standard normal distribution.

$$
\begin{aligned}
\Pr(X \geq 1) &= \int_1^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}x^2} dx \\
&= \lim_{r\to\infty} (F(r) - F(1)) \\
&\approx \lim_{r\to\infty} (F(r)) - (\frac{1}{2} + \frac{1}{\sqrt{2\pi}}(1 - \frac{1^3}{6} + \frac{1^5}{40}) - \frac{1^7}{336}) \\
&\approx 0.5 - .34332 \\
&= .15668
\end{aligned}
$$