12.4: Exponential and normal random variables

Exponential density function

Given a positive constant $k > 0$, the exponential density function (with parameter $k$) is

$$f(x) = \begin{cases} ke^{-kx} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Expected value of an exponential random variable

Let $X$ be a continuous random variable with an exponential density function with parameter $k$.

Integrating by parts with $u = kx$ and $dv = e^{-kx}dx$ so that $du = kdx$ and $v = \frac{1}{k}e^{-kx}$, we find

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

$$= \int_{0}^{\infty} kxe^{-kx}dx$$

$$= \lim_{r \to \infty} \left[ -xe^{-kx} - \frac{1}{k}e^{-kx} \right]_{0}^{r}$$

$$= \frac{1}{k}$$
Variance of exponential random variables

Integrating by parts with $u = kx^2$ and $dv = e^{-kx} dx$ so that $du = 2kx dx$ and $v = \frac{1}{k} e^{-kx}$, we have

\[
\int_0^\infty x^2 e^{-kx} dx = \lim_{r \to \infty} \left( \left[-x^2 e^{-kx}\right]_0^r + 2 \int_0^r xe^{-kx} dx \right)
\]

\[
= \lim_{r \to \infty} \left( \left[-x^2 e^{-kx} - \frac{2}{k} xe^{-kx} - \frac{2}{k^2} e^{-kx}\right]_0^r \right)
\]

\[
= \frac{2}{k^2}
\]

So, $\text{Var}(X) = \frac{2}{k^2} - E(X)^2 = \frac{2}{k^2} - \frac{1}{k^2} = \frac{1}{k^2}$.

Example

Exponential random variables (sometimes) give good models for the time to failure of mechanical devices. For example, we might measure the number of miles traveled by a given car before its transmission ceases to function. Suppose that this distribution is governed by the exponential distribution with mean 100,000. What is the probability that a car’s transmission will fail during its first 50,000 miles of operation?
Solution

\[
\Pr(X \leq 50,000) = \int_{0}^{50,000} \frac{1}{100,000} e^{-\frac{x}{100,000}} dx
\]

\[
= -e^{-\frac{x}{100,000}} \bigg|_{0}^{50,000}
\]

\[
= 1 - e^{-\frac{1}{2}}
\]

\[
\approx .3934693403
\]

Normal distributions

The normal density function with mean \( \mu \) and standard deviation \( \sigma \) is

\[
f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2}
\]

As suggested, if \( X \) has this density, then \( E(X) = \mu \) and \( \text{Var}(X) = \sigma^2 \).

The standard normal density function is the normal density function with \( \mu = \sigma = 1 \). That is,

\[
g(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2}
\]
Taylor expansion for the normal cumulative distribution function

Let \( f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \) be the standard normal density function and let \( F(x) = \int_{-\infty}^{x} f(t)dt \) be the standard normal cumulative distribution function.

We compute a Taylor series expansion,

\[
G(x) = \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \\
= \frac{1}{\sqrt{2\pi}} \int \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n} x^{2n} dx \\
= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n (2n+1)} x^{2n+1} \\
= \frac{1}{\sqrt{2\pi}} \left( x - \frac{1}{6} x^3 + \frac{1}{40} x^5 - \frac{1}{336} x^7 + \cdots \right)
\]

So \( F(x) = G(x) + C \) for some \( C \). As 0 is the expected value, we need \( \frac{1}{2} = F(0) = G(0) + C = C \).
Example

If the continuous random variable $X$ is normally distributed, what is the probability that it takes on a value of more than a standard deviations above the mean?

Solution

Via a change of variables, we may suppose that $X$ is normally distributed with respect to the standard normal distribution. Let $F$ be the cumulative distribution function for the standard normal distribution.

$$\Pr(X \geq 1) = \int_1^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx$$

$$= \lim_{r \to \infty} (F(r) - F(1))$$

$$\approx \lim_{r \to \infty} (F(r)) - \left(\frac{1}{2} + \frac{1}{\sqrt{2\pi}} \left(1 - \frac{1^3}{6} + \frac{1^5}{40} - \frac{1^7}{336}\right)\right)$$

$$\approx 0.5 - 0.34332$$

$$= 0.15668$$