12.3: Expected Value and Variance

If $X$ is a random variable with corresponding probability density function $f(x)$, then we define the expected value of $X$ to be

$$E(X) := \int_{-\infty}^{\infty} x f(x) dx$$

We define the variance of $X$ to be

$$\text{Var}(X) := \int_{-\infty}^{\infty} [x - E(X)]^2 f(x) dx$$

Alternate formula for the variance

As with the variance of a discrete random variable, there is a simpler formula for the variance.
\begin{align*}
\text{Var}(X) &= \int_{-\infty}^{\infty} [x - E(X)]f(x)dx \\
&= \int_{-\infty}^{\infty} [x^2 - 2xE(X) + E(X)^2]f(x)dx \\
&= \int_{-\infty}^{\infty} x^2f(x)dx - 2E(X)\int_{-\infty}^{\infty} xf(x)dx \\
&\quad + E(X)^2\int_{-\infty}^{\infty} f(x)dx \\
&= \int_{-\infty}^{\infty} x^2f(x)dx - 2E(X)E(X) + E(X)^2 \times 1 \\
&= \int_{-\infty}^{\infty} x^2f(x)dx - E(X)^2
\end{align*}

Interpretation of the expected value and the variance

The expected value should be regarded as the average value. When \( X \) is a discrete random variable, then the expected value of \( X \) is precisely the mean of the corresponding data.

The variance should be regarded as (something like) the average of the difference of the actual values from the average. A larger variance indicates a wider spread of values.

As with discrete random variables, sometimes one uses the standard deviation, \( \sigma = \sqrt{\text{Var}(X)} \), to measure the spread of the distribution instead.
Example

The uniform distribution on the interval $[0, 1]$ has the probability density function

$$f(x) = \begin{cases} 
0 & \text{if } x < 0 \text{ or } x > 1 \\
1 & \text{if } 0 \leq x \leq 1 
\end{cases}$$

Letting $X$ be the associated random variable, compute $E(X)$ and $\text{Var}(X)$.

Solution

$$E(X) = \int_{-\infty}^{\infty} x f(x) \, dx$$

$$= \int_{-\infty}^{0} x \times 0 \, dx + \int_{0}^{1} x \times 1 \, dx + \int_{1}^{\infty} x \times 0 \, dx$$

$$= 0 + \frac{1}{2} \left[ x^2 \right]_{0}^{1} + 0$$

$$= \frac{1}{2}$$
Solution, continued

We compute

\[ \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{0}^{1} x^2 dx \]
\[ = \frac{1}{3} x^3 \bigg|_{x=1}^{x=0} \]
\[ = \frac{1}{3} \]

Solution, completed

Hence,

\[ \text{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x)dx - E(X)^2 \]
\[ = \frac{1}{3} - \frac{1}{4} \]
\[ = \frac{1}{12} \]
Another example

Let $X$ be the random variable with probability density function

$$f(x) = \begin{cases} 
  e^x & \text{if } x \leq 0 \\
  0 & \text{if } x > 0
\end{cases}.$$ 

Compute $E(X)$ and $\text{Var}(X)$.

Solution

Integrating by parts with $u = x$ and $dv = e^x dx$, we see that

$$\int x e^x dx = x e^x - e^x + C.$$ 

Thus,

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_{-\infty}^{0} x e^x dx = \lim_{r \to -\infty} \int_{r}^{0} x e^x dx = \lim_{r \to -\infty} \left[ -1 - re^r + e^r \right] = 1.$$ 

[We used L'Hôpital's rule to see that

$$\lim_{r \to -\infty} re^r = \lim_{r \to -\infty} e^r = \lim_{r \to -\infty} \frac{1}{e^{\frac{1}{r}}} = 0.$$]
Solution, continued

We compute

\[
\int x^2 e^x \, dx = x^2 e^x - 2 \int xe^x \, dx = x^2 e^x - 2xe^x + 2e^x + C
\]

So,

\[
\int_{-\infty}^{\infty} x^2 f(x) \, dx = \int_{-\infty}^{0} x^2 e^x \, dx = \lim_{r \to -\infty} (2 - r^2 e^r + 2re^r - 2e^r) = 2
\]

This gives \( \text{Var}(X) = 2 - 1^2 = 1. \)

One more example

Suppose that the random variable \( X \) has a cumulative distribution function

\[
F(x) = \begin{cases} 
\sin(x) & \text{if } 0 \leq x \leq \frac{\pi}{2} \\
0 & \text{if } x < 0 \text{ or } x > \frac{\pi}{2}
\end{cases}
\]

Compute \( E(X) \) and \( \text{Var}(X) \).
Solution

First, we must find the probability density function of $X$. Differentiating we find that the function

$$f(x) = \begin{cases} 
\cos(x) & \text{if } 0 \leq x \leq \frac{\pi}{2} \\
0 & \text{otherwise}
\end{cases}$$

is the derivative of $F$ at all but two points. Thus, $f(x)$ is a probability density function for $X$.

Solution, continued

$$E(X) = \int_{-\infty}^{\infty} x f(x) \, dx$$

$$= \int_{0}^{\frac{\pi}{2}} x \cos(x) \, dx$$

$$= (x \sin(x) + \cos(x)) \bigg|_{x=0}^{x=\frac{\pi}{2}}$$

$$= \frac{\pi}{2} - 1$$
Solution, finished

Integrating by parts, we compute

\[
\text{Var}(X) = \int_0^{\frac{\pi}{2}} x^2 \cos(x)dx - E(X)^2
\]

\[
= (x^2 \sin(x) - 2x \sin(x) + 2x \cos(x)) \bigg|_{x=0}^{x=\frac{\pi}{2}} - \left(\frac{\pi}{2} - 1\right)^2
\]

\[
= \frac{\pi^2}{4} - 2 - \left(\frac{\pi^2}{4} - \pi + 1\right)
\]

\[
= \pi - 3
\]