Section 7.4: Lagrange Multipliers and Constrained Optimization

A constrained optimization problem is a problem of the form *maximize (or minimize) the function* $F(x, y)$ *subject to the condition* $g(x, y) = 0$.

From two to one

In some cases one can solve for y as a function of x and then find the extrema of a one variable function.

That is, if the equation $g(x, y) = 0$ is equivalent to $y = h(x)$, then we may set $f(x) = F(x, h(x))$ and then find the values $x = a$ for which f achieves an extremum. The extrema of F are at $(a, h(a))$.

Example

Find the extrema of $F(x, y) = x^2 y - \ln(x)$ subject to $0 = g(x, y) := 8x + 3y$.

Solution

We solve $y = \frac{-8}{3}x$. Set $f(x) = F(x, \frac{-8}{3}x) = \frac{-8}{3}x^3 - \ln(x)$. Differentiating we have $f'(x) = -8x^2 - \frac{1}{x}$. Setting $f'(x) = 0$, we must solve $x^3 = \frac{-1}{8}$, or $x = \frac{-1}{2}$. Differentiating again, $f''(x) = -16x + \frac{1}{x^2}$ so that $f''(\frac{-1}{2}) = 12 > 0$ which shows that $\frac{-1}{2}$ is a relative minimum of f and $(\frac{-1}{2}, \frac{4}{3})$ is a relative minimum of F subject to $g(x, y) = 0$.

A more complicated example

Find the extrema of $F(x, y) = 2y + x$ subject to
$0 = g(x, y) = y^2 + xy - 1$.

Solution: Direct, but messy

Using the quadratic formula, we find

$$y = \frac{1}{2}(-x \pm \sqrt{x^2 + 4})$$

Substituting the above expression for y in $F(x, y)$ we must find the extrema of

$$f(x) = \sqrt{x^2 + 4}$$

and

$$\varphi(x) = -\sqrt{x^2 + 4}$$

Solution, continued

$$f'(x) = \frac{x}{\sqrt{x^2+4}}$$

and

$$\varphi'(x) = \frac{-x}{\sqrt{x^2+4}}$$

Setting $f'(x) = 0$ (respectively, $\varphi'(x) = 0$) we find $x = 0$ in each case. So the potential extrema are $(0, 1)$ and $(0, -1)$.

Solution, continued

$$f''(x) = \frac{4}{(\sqrt{x^2+4})^3}$$

and

$$\varphi''(x) = \frac{-4}{(\sqrt{x^2+4})^3}$$

Evaluating at $x = 0$, we see that $f''(0) > 0$ so that $(0, 1)$ is a relative minimum and as $\varphi''(0) < 0$, $(0, -1)$ is a relative maximum. (even though $F(0, 1) = 2 > -2 = F(0, -1)$!)

Lagrange multipliers

If $F(x,y)$ is a (sufficiently smooth) function in two variables and $g(x,y)$ is another function in two variables, and we define $H(x,y,z) := F(x,y) + zg(x,y)$, and (a,b) is a relative extremum of F subject to $g(x,y) = 0$, then there is some value $z = \lambda$ such that $\frac{\partial H}{\partial x}|_{(a,b,\lambda)} = \frac{\partial H}{\partial y}|_{(a,b,\lambda)} = \frac{\partial H}{\partial z}|_{(a,b,\lambda)} = 0$.

Example of use of Lagrange multipliers

Find the extrema of the function $F(x,y) = 2y + x$ subject to the constraint $0 = g(x,y) = y^2 + xy - 1$.

Solution

Set $H(x, y, z) = F(x, y) + zg(x, y)$. Then

$$\frac{\partial H}{\partial x} = 1 + zy$$
$$\frac{\partial H}{\partial y} = 2 + 2zy + zx$$
$$\frac{\partial H}{\partial z} = y^2 + xy - 1$$

Solution, continued

Setting these equal to zero, we see from the third equation that $y \neq 0$, and from the first equation that $z = \frac{-1}{y}$, so that from the second equation $0 = \frac{-x}{y}$ implying that $x = 0$. From the third equation, we obtain $y = \pm 1$.

Another Example

Find the potential extrema of the function $f(x,y) = x^2 + 3xy + y^2 - x + 3y$ subject to the constraint that $0 = g(x,y) = x^2 - y^2 + 1$.

Solution

Set $F(x,y,\lambda) := f(x,y) + \lambda g(x,y)$. Then

$$\frac{\partial F}{\partial x} = 2x + 3y - 1 + 2\lambda x \quad (1)$$

$$\frac{\partial F}{\partial y} = 3x + 2y + 3 + 2\lambda y \quad (2)$$

$$\frac{\partial F}{\partial \lambda} = x^2 - y^2 + 1 \quad (3)$$

Solution, continued

Set these all equal to zero.

Multiplying the first line by y and the second by x we obtain:

$$\begin{aligned} 0 &= 2xy + 3y^2 - y + 2\lambda xy \\ 0 &= 2xy + 3x^2 + 3x + \lambda xy \end{aligned}$$

Subtracting, we have

$$0 = 3(x^2 - y^2) + 3x - y$$

Solution, continued

As $0 = x^2 - y^2 + 1$, we conclude that $y = 1 - 3x$. Substituting, we have
$0 = x^2-(1-3x)^2+1 = x^2-9x^2+6x-1+1 = -8x^2+6x = x(6-8x)$.

So the potential extrema are at $(0, 1)$ or $(\frac{3}{4}, \frac{-1}{4})$.