11.4: Infinite series with positive terms

There are several tests for the convergence or divergence of infinite series with all positive terms. We consider two.

- Integral test
- Comparison test

Integral test

If f is a function with $f(x)$ a *decreasing* continuous function defined for all numbers $x \geq k$, then the infinite series $\sum_{n=k}^{\infty} f(n)$ converges if and only if the integral $\int_1^{\infty} f(x)dx$ converges.

Example

Use the integral test to determine whether or not $\sum_{n=1}^{\infty} \frac{1}{n}$ converges.

Solution

Indeed, it does not as

$$\begin{aligned} \int_1^{\infty} \frac{dx}{x} &= \lim_{r\to\infty} \int_1^{r} \frac{dx}{x} \\ &= \lim_{r\to\infty} \ln(x)|_{x=1}^{x=r} \\ &= \lim_{r\to\infty} \ln(r) \\ &= \infty \end{aligned}$$

Another example

Does the series $\sum_{n=1}^{\infty} \frac{n}{e^n}$ converge?

Solution

Consider $f(x) = xe^{-x}$. We compute $f'(x) = (1-x)e^{-x}$ which is negative for all $x > 1$. Thus, f is decreasing.

We compute using integration by parts with $u = x$ so that $du = dx$ and $dv = e^{-x}$ so that $v = -e^{-x}$,

$$\begin{aligned} \int_1^{\infty} xe^{-x}dx &= \lim_{r\to\infty} (-xe^{-x} - e^{-x})|_{x=1}^{x=r} \\ &= \lim_{r\to\infty} -(r+1)e^{-r} + \frac{2}{e} \\ &= \frac{2}{e} \end{aligned}$$

Hence, $\sum_{n=1}^{\infty} \frac{n}{e^n}$ converges.

Comparison tests

If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two sequences of positive numbers for which $a_i \leq b_i$ for every i, then

$$\text{if } \sum_{n=1}^{\infty} a_n \text{ diverges, so does } \sum_{n=1}^{\infty} b_n$$

while

$$\text{if } \sum_{n=1}^{\infty} b_n \text{ converges, so does } \sum_{n=1}^{\infty} a_n$$

moreover,

$0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

Examples

Does the series $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converge?

Solution

Yes: $0 < \frac{1}{n2^n} \leq \frac{1}{2^n}$ for every n. We know $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. Hence, $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converges and is at most 1.