11.1: Taylor polynomials

The derivative as the first Taylor polynomial

If \( f(x) \) is differentiable at \( a \), then the function \( p(x) = b + m(x - a) \) where \( b = f(0) \) and \( m = f'(x) \) is the “best” linear approximation to \( f \) near \( a \).

For \( x \approx a \) we have \( f(x) \approx p(x) \).

Note that \( f(a) = b = p(a) \) and \( f'(a) = m = p'(a) \).

Higher degree Taylor polynomials

If \( f(x) \) is a function which is \( n \) times differentiable at \( a \), then the \( n^{\text{th}} \) Taylor polynomial of \( f \) at \( a \) is the polynomial \( p(x) \) of degree (at most \( n \)) for which \( f^{(i)}(a) = p^{(i)}(a) \) for all \( i \leq n \).
Example

Compute the third Taylor polynomial of $f(x) = e^x$ at $a = 0$.

Solution

Write $p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$. We need to find $c_0$, $c_1$, $c_2$, and $c_3$ so that $p^{(i)}(0) = f^{(i)}(0)$ for $i = 0, 1, 2, \text{ and } 3$.

In our case $f^{(i)}(x) = e^x$ for all $i \geq 0$ and $e^0 = 1$. So, $f^{(i)}(0) = 1$ for all $i$.

We compute $p'(x) = c_1 + 2c_2 x + 3c_3 x^2$, $p''(x) = 2c_2 + 6c_3 x$, and $p'''(x) = 6c_3$. Thus,

$1 = f^{(0)}(0) = p^{(0)}(0) = c_0$.

$1 = f^{(1)}(0) = p^{(1)}(0) = c_1$.

$1 = f^{(2)}(0) = p^{(2)}(0) = 2c_2$ so that $c_2 = \frac{1}{2}$.

Finally, $1 = f^{(3)}(0) = p^{(3)}(0) = 6c_3$ so that $c_3 = \frac{1}{6}$.

Thus, the third Taylor polynomial of $f(x) = e^x$ at $a = 0$ is $p(x) = \frac{1}{6} x^3 + \frac{1}{2} x^2 + x + 1$. 
Another Example

Find the third Taylor polynomial of \( f(x) = \ln(x) \) at \( a = 1 \).

A solution

As before, we write \( p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \) and we find \( p'(x) = c_1 + 2c_2 x + 3c_3 x^2 \), \( p''(x) = 2c_2 + 6c_3 x \), and \( p'''(x) = 6c_3 \). Differentiating, \( f'(x) = \frac{1}{x} = x^{-1} \), \( f''(x) = -x^{-2} \), and \( f'''(x) = 2x^{-3} \). Thus, 

\[
egin{align*}
0 &= f^{(0)}(1) = p^{(0)}(1) = c_0 + c_1 + c_2 + c_3 \\
1 &= f^{(1)}(1) = p^{(1)}(1) = c_1 + 2c_2 + 3c_3 \\
-1 &= f^{(2)}(1) = p^{(2)}(1) = 2c_2 + 6c_3 \\
2 &= f^{(3)}(1) = p^{(3)}(1) = 6c_3
\end{align*}
\]

Solving these equations, we find \( c_3 = \frac{1}{3}, c_2 = -\frac{3}{2}, c_1 = 3, \) and \( c_0 = -\frac{11}{6} \).

That is, the third Taylor polynomial of \( \ln(x) \) at \( a = 1 \) is

\[
\frac{1}{3} x^3 - \frac{3}{2} x^2 + 3x - \frac{11}{6}.
\]
Another solution

We may write any polynomial of degree three as
\[ p(x) = d_0 + d_1(x - 1) + d_2(x - 1)^2 + d_3(x - 1)^3. \]

Differentiating, we have
\[ p'(x) = d_1 + 2d_2(x - 1) + 3d_3(x - 1)^2, \]
\[ p''(x) = 2d_2 + 6d_3(x - 1), \]
and \[ p'''(x) = 6d_3. \]

So, \( p(1) = d_0, \ p'(1) = d_1, \ p''(1) = 2d_2, \) and \( p'''(1) = 6d_3. \)

Hence, if \( p \) is the third Taylor polynomial of \( \ln(x) \) at \( a = 1 \), we have \( d_0 = 0, \ d_1 = 1, \ d_2 = -\frac{1}{2}, \) and \( d_3 = \frac{1}{3}. \)

That is, the third Taylor polynomial of \( \ln(x) \) at \( a = 0 \) is
\[ \frac{1}{3}(x - 1)^3 - \frac{1}{2}(x - 1)^2 + (x - 1). \]

General formula for Taylor polynomials

If we write \( p(x) = \sum_{i=0}^{n} d_i(x - a)^i \), then
\[ p^{(j)}(x) = \sum_{i=j}^{n} \frac{i!}{(i-j)!} d_i(x - a)^{i-j} \]
where \( i! = i \cdot (i - 1) \cdot (i - 2) \ldots 2 \cdot 1. \)

(We define \( 0! = 1 \) and \( (i + 1)! = (i + 1) \cdot i! \).)

In particular, \( p^{(j)}(a) = j!d_j. \) So, if \( p \) is the \( n \)th Taylor polynomial of \( f \) at \( a \), we have \( j!d_j = p^{(j)}(a) = f^{(j)}(a). \)

Thus, \( d_j = \frac{1}{j!}f^{(j)}(a) \) or to put it another way, the \( n \)th Taylor polynomial of \( f \) at \( a \) is \( \sum_{j=0}^{n} \frac{1}{j!}f^{(j)}(a)(x - a)^j. \)
Example

Compute the fifth Taylor polynomial of $f(x) = \sin(x)$ at $a = 0$.

Solution

We compute $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$, $f'''(x) = -\cos(x)$, $f^{(4)}(x) = \sin(x)$, and $f^{(5)}(x) = \cos(x)$. Thus, $f^{(0)}(0) = 0$, $f^{(1)}(0) = 1$, $f^{(2)}(0) = 0$, $f^{(3)}(0) = -1$, $f^{(4)}(0) = 0$, and $f^{(5)}(0) = 1$.

We compute the first few factorials: $0! = 1$, $1! = 1 \cdot 0! = 1 \cdot 1 = 1$, $2! = 2 \cdot 1! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2! = 3 \cdot 2 = 6$, $4! = 4 \cdot 3! = 4 \cdot 6 = 24$, and $5! = 5 \cdot 4! = 5 \cdot 24 = 120$.

Therefore, the fifth Taylor polynomial of $f(x) = \sin(x)$ at $a = 0$ is

$$\frac{1}{120} x^5 - \frac{1}{6} x^3 + x.$$
Error estimates

If \( f(x) \) is \((n+1)\) times differentiable between on the interval \([a,x]\) (or \([x,a]\) if \(x < a\)) and \(p(x)\) is the \(n^{th}\) Taylor polynomial of \(f\) at \(a\), then there is a number \(a \leq c \leq x\) so that

\[
f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}.
\]

So, if we can find \(M\) so that \( |f^{(n+1)}(y)| \leq M \) whenever \(a \leq y \leq x\), we would know that \( |f(x) - p(x)| \leq \frac{M}{(n+1)!} (x-a)^{n+1} \).

Example

Find a decimal approximation to \(e\) valid to the hundredths place.
Solution

We will find $n$ so that if $p(x)$ is the $n^{th}$ Taylor polynomial for $f(x) = e^x$ at $a = 0$, then $|e - p(1)| = |f(1) - p(1)| < \frac{1}{200}$.

We know that $f^{(n+1)}(x) = e^x$ and that on the interval from zero to one this function is bounded by 3.

Thus, $|e - p(1)| \leq \frac{3}{(n+1)!} (1 - 0)^{n+1} = \frac{3}{(n+1)!}$.

So, we want $n$ so that $\frac{3}{(n+1)!} < \frac{1}{120}$ or what is the same thing $(n + 1)! > 360$. If $n = 5$, then $(n + 1)! = 6! = 720 > 360$.

Now, $p(x) = \frac{1}{120} x^5 + \frac{1}{24} x^4 + \frac{1}{6} x^3 + \frac{1}{2} x^2 + x + 1$. So, $e \approx \frac{1}{120} + \frac{1}{24} + \frac{1}{6} + \frac{1}{2} + 1 + 1 = \frac{1+5+20+60+120+120}{120} = \frac{326}{120} = \frac{163}{60} \approx 2.7166666666......$

A better approximation to $e$

In point of fact, $e \approx 2.718281828459045235360287471$

35266249775724709369995957 4966967627724076630353547
59457138217852516642742 7466391932003 059921817413
566629043572900334295 2605956307 38132328 6279434907632
338298807531952 5101901157 38341879307021540891
499348841675092447614 606880822648001 68477411853742
3454424371075 39077744999206955170 276183860626133
138458300075 204493382656 0297606737113 200793287091
27443747047 2306969772093 101416928368 190255151
086574637721 1125238978442 505695369677 078544996996794686
445490598793 163688923009879312773617821 54249992295763
514822082698951 93668033182528 869398496 465105820939239
829488793320362 5094431.