A differential equation
\[ y' = f(y, t) \]
may be approximated as a difference equation. If \( \Delta t \approx 0 \), then
\[
y(a + \Delta t) \approx y(a) + y'(a)\Delta t = y(a) + f(y(a), a)\Delta t
\]

Euler’s Method
Iterating the approximation \( y(a + \Delta t) \approx y(a) + f(y(a), a)\Delta t \), we can numerically approximate solutions to initial value problems \( y' = f(y, t) \) and \( y(t_0) = y_0 \).

That is, given that \( y \) satisfies the above initial value problem, to approximate \( y(a) \), fix a positive integer \( n \), set \( \Delta = \frac{a-t_0}{n} \), and define \( t_i := t_0 + i\Delta \) (for \( 0 \leq i \leq n \)).
Euler’s method, continued

We know that \( y(t_0) = y_0 \). Approximating, we have

\[
\begin{align*}
y(t_1) &= y(t_0 + \Delta) \\
&\approx y(t_0) + \Delta y'(t_0) \\
&= y_0 + \Delta f(y_0, t_0) \\
&=: y_1
\end{align*}
\]

Repeating this process, we find that \( y(t_2) \approx y_1 + \Delta f(y_1, t_1) =: y_2, \ldots, y(a) = y(t_n) \approx y_{n-1} + \Delta f(y_{n-1}, t_{n-1}). \)

Example

Approximate the value of \( y(1) \) when \( y' = ty + 1 \) and \( y(0) = 0 \) using \( n = 2 \).
Solution

Note that a symbolically solve $y' = ty + 1$ one must find an antiderivative to $e^{-\frac{1}{2}t^2}$.

Here $\Delta = \frac{1-0}{2} = 0.5$.

We compute

$$y(0.5) \approx 0 + (0.5)(0(0) + 1)$$
$$= 0.5$$
$$= y_1$$

$$y(1) \approx 0.5 + (0.5)((0.5)(0.5) + 1)$$
$$= 0.5 + 0.5(1.25)$$
$$= 0.5 + 0.625$$
$$= 1.125$$

Another Example

Approximate $y(1)$ when $y' = \sin(y)$ and $y(0) = .1$ using $n = 5$ subdivisions.
Obstructions to symbolic solutions

This time, our symbolic methods fail twice! To use the method of separation of variables, we would need to find an antiderivative of \( \csc(y) \). Even if we were to succeed with this step, we would have to invert the function \( \int \csc(y)\,dy \).

Solution

In this case, we compute mechanically.
\[
\Delta = \frac{1-0}{2} = 0.2, \quad y_0 = 0.1, \quad \text{and we wish to find } y_5 \approx y(1).
\]

\[
y(2) \approx y_0 + \Delta \sin(y_0)
= 0.1 + (0.2) \sin(0.1)
\approx 0.1000 + (0.2000)(0.0998)
\approx 0.1200
=: y_1
\]
\[
y(.4) \approx y_1 + \Delta \sin(y_1) \\
= 0.1200 + (0.2000) \sin(0.1200) \\
\approx 0.1429 \\
=: y_2
\]

\[
y(.6) \approx y_2 + \Delta \sin(y_2) \\
= 0.1429 + (0.2) \sin(0.1429) \\
\approx .1714 \\
=: y_3
\]

\[
y(.8) \approx y_3 + \Delta \sin(y_3) \\
= .1714 + (0.2) \sin(.1714) \\
\approx .2055 \\
=: y_4
\]

\[
y(1) \approx y_4 + \Delta \sin(y_4) \\
= .2055 + (0.2) \sin(.2055) \\
\approx .2463
\]