Section 10.4: Applications of Differential Equations

In general, one uses differential equations (and the methods we have developed for their solution) when a function is described by conditions on its rate of change, but one wishes to find a closed form expression for the function.

Free fall

An object falling in a vacuum subject to a constant gravitational force accelerates at a constant rate.

If the object were to be dropped from rest and to attain a velocity of $5m/s$ after one second, how fast would it be traveling after five seconds?

Solution

Let $v(t)$ be the velocity at time t seconds measured in meters per second. Then we know that $v(0) = 0$, that $v(1) = 5$, and that $v'' = 0$ (the acceleration, the rate of change of the velocity, so v', is constant).

Integrating the equation $v'' = 0$ with respect to t, we see that $v'(t) - v'(0) = 0$. Thus, if $C_1 = v'(0)$, we have $v'(t) = C_1$. Integrating again, we see that $v(t) - v(0) = C_1 t$. Setting $C_2 := v(0)$, we have $v(t) = C_2 + C_1 t$.

Evaluating at 0 and 1 we have $0 = v(0) = C_2 + C_1(0) = C_2$ and $5 = v(1) = C_2 + C_1(1) = 0 + C_1 = C_1$. Thus, $v(t) = 5t$ so that $v(5) = 25$.

Free fall with air resistance

An object performing a free fall subject to a constant gravitational force in a viscous fluid is slowed by a drag which is proportional to its velocity.

Find a general expression for the velocity of such an object.

Solution

Let α be the constant rate of gravitational acceleration, μ the constant of proportionality for the drag force, and v_0 the initial velocity.

Then the velocity, v, satisfies

$$\begin{aligned} \text{acceleration} &= [\text{ constant gravitational acceleration }] \\ &\quad -[\text{ drag }] \\ &= [\text{ constant gravitational acceleration }] \\ &\quad -[\text{ a quantity proportional to the velocity }] \end{aligned}$$

Solution, continued

In symbols, $v' = \alpha - \mu v$ or $v' + \mu v = \alpha$.

This is a linear first order differential equation which we may solve using the method integration factors.

Here $A(t) = \mu t$ and $\int_0^T e^{\mu t} \alpha dt = \frac{\alpha}{\mu}(e^{\mu T} - 1)$.

So, $v(t) = v_0 e^{-\mu t} + \frac{\alpha}{\mu}(1 - e^{-\mu t})$.

Loan repayment

A loan has a fixed interest rate of 5 % (the interest is compounded continuously) and the borrower repays the loan at a constant rate of $10,000 dollars per year. If the initial value of the loan was $100,000, when will the debt be retired?

Solution

Let $y(t)$ be the remaining principal at time t years. We were told that $y(0) = 100,000$ and we wish to find t so that $y(t) = 0$.

$$\begin{aligned}\text{change of the principal} \quad &= \quad [\text{ rate of new debt due to interest }] \\ &\quad\;\; -[\text{ rate at which the debt is paid }]\end{aligned}$$

Solution, continued

In symbols, $y' = (0.05)y - 10,000$ or $y' - (0.05)y = -10,000$.

As $\frac{d}{dt}(-(0.05)t) = -0.05$, we see that
$y(t) = 200,000 - 100,000e^{0.05t}$.

So, $y(t) = 0$ when $e^{0.05t} = 2$ or $\frac{t}{20} = 20\ln(2) \approx 13.86$.