Section 10.2: Separation of variables

The method of separation of variables applies to differential equations of the form

\[ y' = p(t)q(y) \]

where \( p(t) \) and \( q(x) \) are functions of a single variable.

Example

Find the general solution to the differential equation

\[ y' = ty^2 \]
Solution

Any constant solution to this equation would have $0 \equiv ty^2$ so that $y \equiv 0$.

Avoiding the constant solution, we may divide both sides of the equation by $y^2$ and then we solve:

\[
\frac{T^2}{2} = \int_0^T t \, dt = \int_0^T \frac{y'}{y^2} \, dt = \int_{y(0)}^{y(T)} y^{-2} \, dy = \frac{-1}{y(T)} \frac{y(T)}{y(0)} = \frac{1}{y(0)} - \frac{1}{y(T)}
\]

So, if we set $C := y(0)$, we have $y = \frac{2C}{t - C(t)}$. 
General procedure

To solve the differential equation $y' = p(t)q(y)$:

- Find the constant solutions by solving for $q(c) = 0$.
- Find $P(t)$ an antiderivative of $p(t)$ and $Q(x)$ an antiderivative of $\frac{1}{q(x)}$.
- Write $c = y(0)$. We find $Q(y) - Q(c) = \int_{y}^{c} \frac{dy}{q(y)} = \int_{0}^{T} p(t)dt = P(T) - P(0)$.
- Solve for $y$.

Example

Find the general solution of $y' = y \sin(t) - \sin(t)$
Solution

We begin by rewriting the equation at \( y' = (y - 1) \sin(t) \).

The only constant solution is \( y \equiv 1 \).

Integrating, we find that \( \ln(|y - 1|) \) is an antiderivative of \( \frac{1}{y - 1} \)
while \( -\cos(t) \) is an antiderivative of \( \sin(t) \).

Let \( C = y(0) \). Then we have \( \ln(|y - 1|) - \ln(|C - 1|) = 1 - \cos(t) \).

Adding \( \ln(|C - 1|) \) to both sides and applying the exponential function, we conclude that \( |y - 1| = |C - 1| e^{1 - \cos(t)} \).

As the solution \( y \) must be continuous, the signs of \( y - 1 \) and \( C - 1 \) agree. Thus, \( y = 1 + (C - 1) e^{1 - \cos(t)} \).

Note: In this case the constant solution has the same form.

Yet another example

Find the general solution to the differential equation \( y' = ty + 1 \)
No elementary solution!
The method of separation of variables does not apply as the function $ty + 1$ cannot be written as the product of a function of $y$ by a function of $t$.

**Scholium:** Using Taylor series expansions (a topic which we shall discuss next month), one can compute an expression for solutions to the equation $y' = ty + 1$.

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**Another Example**

Find the general solution to the equation

$$y' = \frac{\sec^2 t}{y + 1}$$
Solution

There are no constant solutions as \( \frac{1}{x+1} \) is never zero. Note, however, that we cannot have \( y(t) = -1 \) as the differential equation would require \( y \) to be nondifferentiable at such a point.

As before, we set \( C = y(0) \). Multiplying by \( y + 1 \) and integrating, we find

\[
\tan(T) = \int_0^T \sec(t) dt = \int_0^T (y(t) + 1)y'(t) dt = \int_C^y (y + 1) dy = \frac{1}{2} y(T)^2 + y(T) - \frac{1}{2} C^2 - C
\]

Solution, continued

So, \( y \) satisfies the equation

\[
y^2 + 2y - C^2 - 2C - 2 \tan(t) = 0
\]

From the quadratic formula, we compute that

\[
y = \frac{-2 \pm \sqrt{2^2 - 4(-C^2 - 2C - 2 \tan(t))}}{2} = -1 \pm \sqrt{1 + C^2 + 2C + 2 \tan(t)} = -1 \pm \sqrt{(C + 1)^2 + 2 \tan(t)}
\]
Undefined solutions, multiple solutions

• For each choice of $C \neq -1$, we found two solutions to the initial value problem $y' = \frac{\sec^2(t)}{y+1}$, namely
  
  $y = -1 + \sqrt{(C+1)^2 + 2\tan(t)}$ and
  
  $y = -1 - \sqrt{(C+1)^2 + 2\tan(t)}$. However, only
  
  $y = -1 + \sqrt{(C+1)^2 + 2\tan(t)}$ satisfies $y(0) = C$.

• Strictly speaking, there is no solution with $y(0) = -1$, but there is a solution having $\lim_{t \to 0^+} y(t) = -1$.

• No solution to the differential equation is defined for all values of $t$. If $t \ll 0$ so that $\tan(t) < -\frac{(C+1)^2}{2}$, then
  
  $\sqrt{(C+1)^2 + 2\tan(t)}$ is not a real number.

Another Example

Find a function $y$ satisfying $y(0) = 5$ and $y' = \frac{1}{3}$.
Solution

As the exponential function never attains the value zero, there are no constant solutions to this differential equation. Multiplying both sides of the equation by \( e^y \) and integrating, we obtain:

\[
\frac{1}{2} T^2 = \int_0^T t \, dt = \int_0^T e^{y(t)} y'(t) \, dy = \int_{y(0)}^{y(T)} e^y \, dy = e^{y(T)} - e^5
\]

Solution, continued

Adding \( e^5 \) to both sides of this equation and taking the natural logarithm, we compute

\[
y = \ln(e^y) = \ln(e^5 + \frac{1}{2} T^2)
\]