Section 9.4: Approximation of Definite Integrals
Review of Riemann Sums

If $a < b$, $f(x)$ is a function on $[a, b]$, and
$a = x_0 \leq a_0 \leq x_1 \leq a_1 \leq \cdots a_{n-1} \leq x_n = b$, then the Riemann sum associated to these data is

$$\sum_{i=0}^{n-1} f(a_i)(x_{i+1} - x_i)$$

By *definition*, the integral, $\int_a^b f(x)dx$ is the limit (if it exists) of these Riemann sums as maximum of $|x_{i+1} - x_i|$ tends to zero.

Uniform divisions

For the sake of convenience, we often assume that the interval $[a, b]$ has been decompose into N pieces of equal length, for some positive integer N. The length of each piece is then $\Delta := \frac{b-a}{N}$.

So, $x_i = a + i\Delta$ and $x_i \leq a_i \leq x_{i+1}$ and for such a uniform decomposition, the Riemann sum is

$$\begin{aligned} \sum_{i=0}^{N-1} (f(a_i)(x_{i+1} - x_i)) &= \sum_{i=0}^{N-1} (f(a_i)(a + (i+1)\Delta - a - i\Delta)) \\ &= \sum_{i=0}^{N-1} f(a_i)\Delta \end{aligned}$$

Left- and Right-hand rules

Let $a \leq b$, N, and $f(x)$ be given. Then the *lefthand* Riemann sum approximating $\int_a^b f(x)dx$ with N subdivisions is given by setting $a_i := x_i = a + i\Delta$ where $\Delta = \frac{b-a}{N}$.

$$L = \sum_{i=0}^{N-1} f(a + i\Delta)\Delta$$

The *righthand* Riemann sum is given by setting $a_i := x_{i+1} = a + (i+1)\Delta$.

$$R = \sum_{i=0}^{N-1} f(a + (i+1)\Delta)\Delta$$

Example

Compute the left and the right approximations to $\int_1^9 x^2 dx$ with $N = 4$ subdivisions.

Solution

In this case $\Delta = \frac{9-1}{4} = 2$. So,

$$
\begin{aligned}
L &= \sum_{i=0}^{3} (1+i2)^2 2 \\
&= (1^2 + 3^2 + 5^2 + 7^2)2 \\
&= (1 + 9 + 25 + 49)2 \\
&= 168
\end{aligned}
$$

Solution, continued

$$
\begin{aligned}
R &= \sum_{i=0}^{3} (1+(i+1)2)^2 2 \\
&= (3^2 + 5^2 + 7^2 + 9^2)2 \\
&= (9 + 25 + 49 + 81)2 \\
&= 328
\end{aligned}
$$

Midpoint rule

The midpoint approximation (to the integral $\int_a^b f(x)dx$ with N subdivisions) is given by taking a_i to be the midpoint of the interval $[x_i, x_{i+1}]$ where $x_i = a + i\Delta$ and $\Delta = \frac{b-a}{N}$. Thus, $a_i = a + (i + \frac{1}{2})\Delta$ and

$$M = \sum_{i=0}^{N-1} f(a_i)\Delta = \sum_{i=0}^{N-1} f(a + (i + \frac{1}{2})\Delta)\Delta$$

Example

Approximate $\int_1^9 x^2 dx$ using the midpoint rule and $N = 4$ subdivisions.

Solution

$$
\begin{aligned}
M &= \sum_{i=0}^{3}(1+((i+\frac{1}{2})2))^2 2 \\
&= (2^2+4^2+6^2+8^2)2 \\
&= (4+16+36+64)2 \\
&= 240
\end{aligned}
$$

Trapezoidal rule

Rather than approximating the area bounded by a function by rectangles, one may use other shapes. For example, one may use trapezoids. The area of the trapezoid with corners at $(a,0)$, $(b,0)$, $(a,f(a))$, and $(b,f(b))$ is $\frac{1}{2}(f(a)+f(b))(b-a)$.

The trapezoidal approximation to $\int_a^b f(x)dx$ with N subdivisions is

$$T = \sum_{i=0}^{N-1} \frac{1}{2}(f(x_i)+f(x_{i+1}))\Delta$$

where $\Delta = \frac{b-a}{N}$ and $x_i = a + i\Delta$.

Example

Compute the trapezoidal approximation to $\int_1^9 x^2 dx$ with 4 subdivisions.

Solution

$$\begin{aligned} T &= \sum_{i=0}^{3} \frac{1}{2}((1+i2)^2 + (1+(i+1)2)^2)2 \\ &= \frac{1}{2}(1^2 + 3^2 + 3^2 + \\ &\quad 5^2 + 5^2 + 7^2 + 7^2 + 9^2)2 \\ &= (1+9+9+25+25+49+49+81) \\ &= 248 \end{aligned}$$

Other formulae for the trapezoidal rule

$$\begin{aligned}
T &= \sum_{i=0}^{N-1} \frac{1}{2}(f(x_i) + f(x_{i+1})\Delta \\
&= \frac{1}{2}[(\sum_{i=0}^{N-1} f(x_i)\Delta) + \\
&\quad (\sum_{i=0}^{N-1} f(x_{i+1})\Delta)] \\
&= \frac{1}{2}[L + R]
\end{aligned}$$

Another formula for T

$$\begin{aligned}
T &= \sum_{i=0}^{N-1} \frac{1}{2}(f(x_i) + f(x_{i+1}))\Delta \\
&= \frac{1}{2}(f(x_0) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + f(x_3) + \\
&\quad \cdots + f(x_{N-1}) + f(x_N))\Delta \\
&= \frac{1}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + \\
&\quad \cdots + 2f(x_{N-1}) + f(x_N)) \\
&= \frac{1}{2}[f(x_0) + 2(\sum_{i=1}^{N-1} f(x_i)) + f(x_N)]\Delta
\end{aligned}$$

Simpson's Rule

Simpson's rule for approximating integrals is based on approximating $\int_{x_i}^{x_{i+1}} f(x)dx$ by the area bounded by the parabola passing through $(x_i, f(x_i))$, $(\frac{x_i+x_{i+1}}{2}, f(\frac{x_i+x_{i+1}}{2}))$, and $(x_{i+1}, f(x_{i+1}))$.

Rather than deriving Simpson's rule from its geometric description, we write the Simpson's rule approximation in terms of the midpoint and trapezoidal approximations.

$$S = \frac{2}{3}M + \frac{1}{3}T$$

Another formula for Simpson's rule

Directly,

$$\begin{aligned} S &= \frac{1}{6}[f(a) + 4f(a + \frac{1}{2}\Delta) + 2f(a+\Delta) + 4f(a + \frac{3}{2}\Delta) + \\ &\quad \cdots + 4f(a + [N - \frac{1}{2}]\Delta) + f(b)]\Delta \\ &= \frac{1}{6}[f(a) + 4(\sum_{i=0}^{N-1} f(a + (i + \frac{1}{2})\Delta)) \\ &\quad +2(\sum_{i=1}^{N-1} f(a + i\Delta)) + f(b)]\Delta \end{aligned}$$

Example

Compute the Simpson's rule approximation to $\int_1^9 x^2 dx$ with $N = 4$ subdivisions.

Solution

$$
\begin{aligned}
S &= \frac{1}{6}[1^2 + 4(2^2 + 4^2 + 6^2 + 8^2) \\
&\quad +2(3^2 + 5^2 + 7^2) + 9^2]2 \\
&= \frac{1}{3}[1 + 4(4 + 16 + 36 + 64) \\
&\quad +2(9 + 25 + 49) + 81] \\
&= \frac{1}{3}[1 + 4(120) + 2(83) + 81] \\
&= \frac{1}{3}[1 + 480 + 166 + 81] \\
&= \frac{1}{3}[656] \\
&= 242\frac{2}{3}
\end{aligned}
$$

Comparison

The exact value of $\int_1^9 x^2 = \frac{1}{3}x^3|_1^9 = \frac{1}{3}[9^3 - 1^3] = \frac{728}{3} = 242\frac{2}{3}$.

Approximations for $N = 4$.

Left	168
Midpoint rule	240
Simpson's rule	$242\frac{2}{3}$
Exact value	$242\frac{2}{3}$
Trapezoidal rule	248
Right	328

Error Analysis

The error of an approximation α is the absolute value of the difference between α and the exact value of the itegral.

- If $|f''(x)| \leq A$ for all $x \in [a, b]$, then the error of a midpoint approximation with N subdivisions is at most $\frac{A(b-a)^3}{24N^2}$.
- If $|f''(x)| \leq A$ for all $x \in [a, b]$, then the error of a trapezoidal approximation with N subdivisions is at most $\frac{A(b-a)^3}{12N^2}$.
- If $|f''''(x)| \leq A$ for all $x \in [a, b]$, then the error of a Simpson's rule approximation with N subdivisions is at most $\frac{A(b-a)^5}{2880n^4}$.

Example

How many subdivisions do we need to guarantee that a midpoint approximation to $\int_1^2 \frac{dx}{x}$ approximates $\ln(2)$ to two decimal points.

Solution

We need to have an error of less than 0.005. So, we want

$$0.005 > \frac{A(2-1)^2}{24N^2}$$

where $A = \max\{|f''(x)| \mid 1 \leq x \leq 2\}$.

We compute $f'(x) = -x^{-2}$ and $f''(x) = 2x^{-3}$. This function is decreasing between 1 and 2. So, $A = 2$. So, we want $N^2 > \frac{2\times 200}{24} = 16\frac{2}{3}$. Thus, $N = 5$ suffices.

We compute

$$\begin{aligned} \ln(2) &= \int_1^2 \frac{1}{x} dx \\ &\approx (1/(1.1) + 1/(1.3) + 1/(1.5) + 1/(1.7) + 1/(1.9)).2 \\ &= 0.6919\ldots \end{aligned}$$