Section 9.2: Integrations by Parts

A Test Problem

Perform the following indefinite integration.

\[ \int x \sin(x)\,dx \]

Answer

\[ \int x \sin(x)\,dx = \sin(x) - x \cos(x) + C \]
Product Rule Recalled

\[(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)\]

Inverting the Chain Rule: Integration by Substitution

\[\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx\]
Formalism of Integration by Parts

Often, one finds two functions $u$ and $v$ so that the integrand may be written as $udv$ where $dv = v'(x)dx$. If we succeed in so doing, then from the equality

$$u(x)v'(x) = u(x)v(x) - v(x)u'(x)$$

we see that if $f(x) = u(x)v'(x)$, then

$$\int f(x)dx = \int u(x)v'(x)dx = \int u(x)v(x)dx - \int v(x)u'(x)dx$$

If $\int v(x)u'(x)dx$ is easier to evaluate than is $\int f(x)dx$, then the method succeeds.

An integral revisited

Take $u = x$ and $v = -\cos(x)$ so that $dv = \sin(x)dx$ and $du = dx$. Then,

$$\int x \sin(x)dx = \int udv$$

$$= uv - \int vdu$$

$$= -x \cos(x) + \int \cos(x)dx$$

$$= -x \cos(x) + \sin(x) + C$$
Example

Integrate:

\[ \int xe^x \, dx \]

Solution

Take \( u = x \) and \( v = e^x \) so that \( du = dx \) and \( dv = e^x \, dx \).

\[
\int xe^x \, dx = \int udv = uv - \int vdu = xe^x - \int e^x \, dx = xe^x - e^x + C
\]
Integrate

\[ \int e^x \cos(x) \, dx \]

**Solution**

Set \( u = e^x \) and \( v = \sin(x) \) so that \( du = e^x \) and \( dv = \cos(x) \, dx \).

Then

\[
\int e^x \cos(x) \, dx = \int u \, dv = uv - \int v \, du = e^x \sin(x) - \int e^x \sin(x) \, dx = e^x \sin(x) + \int (-e^x \sin(x)) \, dx
\]
Solution, continued

Set \( w = e^x \) and \( y = \cos(x) \) so that \( dw = e^x \, dx \) and \( dy = -\sin(x) \, dx \).

Then

\[
\int -e^x \sin(x) \, dx = \int wdy \\
= wy - \int y \, dw \\
= e^x \cos(x) - \int e^x \cos(x) \, dx
\]

Substituting, we have

\[
\int e^x \cos(x) \, dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x) \, dx
\]

Adding \( \int e^x \cos(x) \, dx \) and dividing by 2 gives

\[
\int e^x \cos(x) \, dx = \frac{e^x}{2} (\sin(x) + \cos(x)) + C
\]
A third example

Integrate

\[ \int \ln(x) \, dx \]

Solution

Set \( u = \ln(x) \) and \( v = x \) so that \( du = \frac{dx}{x} \) and \( dv = dx \).

\[
\int \ln(x) \, dx = \int u \, dv = uv - \int v \, du = x \ln(x) - \int x \cdot \frac{1}{x} \, dx = x \ln(x) - \int dx = x \ln(x) - x + C
\]
A final example

Integrate

\[ \int \frac{xe^{2x}}{x^2 + 4x + 4} \, dx \]

Solution

Note that \( \frac{1}{4x^2 + 4x + 1} = (2x + 1)^{-2} \). Via the substitution \( w = 2x + 1 \) (with \( dw = 2 \, dx \)) we find that

\[ \int \frac{dx}{4x^2 + 4x + 1} = \int (2x + 2)^{-2} \, dx \]

\[ = \frac{1}{2} \int w^{-2} \, dw \]

\[ = -\frac{1}{2} w^{-1} + C \]

\[ = -\frac{1}{4x + 2} + C \]
Solution, continued

Set \( u = xe^{2x} \) and \( v = -\frac{1}{4x+2} \) (so that \( dv = \frac{dx}{4x+2} \) and \( du = e^{2x} + 2xe^{2x} = (2x + 1)e^{2x} \)).

\[
\int \frac{xe^{2x}}{4x^2 + 4x + 1} \, dx = \int udv
\]
\[
= uv - \int vdu
\]
\[
= -\frac{xe^{2x}}{4x + 2} + \int \frac{(2x + 1)e^{2x}}{4x + 2} \, dx
\]
\[
= -\frac{xe^{2x}}{4x + 2} + \frac{1}{2} \int e^{2x} \, dx
\]
\[
= -\frac{xe^{2x}}{4x + 2} + \frac{1}{4} e^{2x} + C
\]