Math 16b: Calculus and Analytic Geometry

http://www.math.berkeley.edu/~scanlon/m16bs04/index.html

Chapter 7: Functions of Several Variables

Section 7.1: Examples of Functions of Several Variables

- $f(x, y) = x + y^2$
- $g(\theta, \varphi) = \sin(4\theta)\cos(\varphi^2)$
- $h(s, t, u) = \frac{e^{st}+u}{t^2+us}$

A function need *not* be expressed in terms of a formula.

- The population P of the state s at the beginning of the year y is a function of the variables s and y.

Traditionally, the variables of a function of several variables are written as x, y, z or if there are more than three variables, using subscripts $x_1, x_2, \ldots, x_n$.

The functions themselves are *usually* written using symbols such a f, g, h, F, G, H and if it is important to list the variables as, *eg* $f(x, y)$, $g(x, y, z)$, or $H(x_1, x_2, x_3, x_4, x_5)$.

Graphing

For function of two variables, one may graph this function by plotting the solutions to $z = f(x, y)$ in three space.

Examples:

- $z = x + y^2$
- $z = \frac{1}{x+y} + y^2$
- $z = \sin(xy^2)$

Level curves

One may produce a two-dimensional graph of a function of two variables by plotting the solutions to $f(x, y) = c$ for various constants c.

A contour map is precisely such a graph. Here the variables are the lattitude and longitude of a point on the Earth and the function gives the altitude.

Functions of several variables in applied problems

Suppose that one wishes to produce a structure in the shape of a rectangular box. The material for the floor costs \$7 per square foot, the material for the walls costs \$5 per square foot, and the material for the roof costs \$2 per square foot. Write the total cost as a function of w, the width, ℓ, the length, and h, the height, of the structure.

Solution

$$\begin{aligned}\text{Area of the floor} &= w\ell \\ \text{Area of walls} &= (2wh) + (2\ell h) \\ \text{Area of roof} &= w\ell\end{aligned}$$

Solution (continued)

$$\begin{aligned}\text{Total cost} &= \text{Floor cost} + \text{Wall cost} + \text{Roof cost}\\ &= 7\text{ Floor area } + 5\text{ Wall area } + 3\text{ Roof area}\\ &= 7w\ell + 5((2wh) + 2(\ell h)) + 2w\ell\\ &= 9w\ell + 10wh + 10\ell h\end{aligned}$$

Section 7.2: Partial Derivatives

If $F(x, y, z)$ is a function of several variables, then for any fixed value of x and y, say, $x = a$ and $y = b$, the function $f(z) := F(a, b, z)$ is a function of the single variable z.

As such, it makes sense to compute the derivative of f, or what is the same thing, the *derivative of f with respect to z*:

$$\frac{\partial F}{\partial z} = f'(z)$$

Derivatives as limits

For a function of one variable, $f(x)$, the derivative of f at a is defined as a limit:

$$f'(a) = \frac{df}{dx}|_{x=a} := \lim_{\epsilon \to 0} \frac{f(a+\epsilon) - f(a)}{\epsilon}$$

Partial derivatives as limits

For a function of several variables, partial derivatives are defined by the same kind of limit.

$$\frac{\partial F}{\partial x}(x, y, z) := \lim_{\epsilon \to 0} \frac{F(x+\epsilon, y, z) - F(x, y, z)}{\epsilon}$$

Computing partial derivatives

In general, to compute the partial derivative of a function with respect to some variable, treat the function as a function of that single variable with all the other named variables regarded as constants.

Computing partial derivatives: Example 1

If C is a constant and n a natural number, then the formula

$$\frac{d}{dx}(Cx^n) = Cnx^{n-1}$$

is familiar to you.

Instead, we could consider this monomial as a function of three variables $f(x, y, z) = yx^z$ (at least for $z \geq 0$) and the above formula expresses

$$\frac{\partial f}{\partial x} = yzx^{z-1}$$

Computing partial derivatives: Example 2

Let $F(x, y, z) = x\sin(y) + z^2$. Compute $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, and $\frac{\partial F}{\partial z}$.

Solution

$$\begin{aligned} \frac{\partial F}{\partial x} &= \sin(y) \\ \frac{\partial F}{\partial y} &= x\cos(y) \\ \frac{\partial F}{\partial z} &= 2z \end{aligned}$$

Computing partial derivatives: Example 3

Let $g(x,y) = xe^{xy^2}$. Compute $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$.

Solution

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{d}{dx}(x) \cdot e^{xy^2} + x\frac{d}{dx}(e^{xy^2}) \\ &= e^{xy^2} + x(y^2 e^{xy^2}) \\ &= (1 + xy^2)e^{xy^2}\end{aligned}$$

Solution continued

$$\begin{aligned}\frac{\partial g}{\partial y} &= \frac{d}{dy}(xe^{xy^2}) \\ &= x\frac{d}{dy}(e^{xy^2}) \\ &= x\frac{d}{dt}(e^t)|_{t=xy^2}\frac{d}{dy}(xy^2) \\ &= xe^{xy^2}(2xy) \\ &= 2x^2ye^{xy^2}\end{aligned}$$

Geometric interpretation

Unlike a curve, a surface has many tangent lines at each point. The partial derivatives give the slopes of the tangent lines at a point in a specific direction.

More precisely, the partial derivative at a point P of a function F with respect to x is the slope of the tangent line to the graph of F at $(P, f(P))$ along the direction where all coordinates save x are held fixed.

Partial derivatives as rates of change

As with derivatives of a function of a single variable, partial derivatives may be interpreted as rates of change. In this case, $\frac{\partial f}{\partial x}$ is the rate at which f changes relative to changes in the x-variable with all other variables held fixed.

Partial derivatives as rates of change: an example

Let $f(x, y) = \frac{x}{y}$. Compute and interpret $\frac{\partial f}{\partial x}|_{(1,2)}$ and $\frac{\partial f}{\partial y}|_{(1,2)}$.

Solution

$\frac{\partial f}{\partial x}|_{(1,2)} = \frac{1}{y}|_{(1,2)} = \frac{1}{2}$. So the slope of the tangent line in the x direction at $(1, 2, \frac{1}{2})$ is $\frac{1}{2}$.

$\frac{\partial f}{\partial y}|_{(1,2)} = \frac{-x}{y^2}|_{(1,2)} = \frac{-1}{4}$. That is, the slope of the tangent line in the y-direction is $\frac{-1}{4}$.

Notice that f is increasing in the x-direction while it is decreasing in the y-direction.

Higher order derivatives

A partial derivative of a function is itself a function and may be differentiated again.

It is a non-trivial, though true, theorem that for a sufficiently smooth function the order of differentiation is immaterial. That is, $\frac{\partial}{\partial x}(\frac{\partial F}{\partial y}) = \frac{\partial}{\partial y}(\frac{\partial F}{\partial x}) =: \frac{\partial^2 F}{\partial x \partial y}$

Second partial derivatives: an example

Let $F(x,y) = x^2y + y^3$. Compute $\frac{\partial^2 F}{\partial x^2}$, $\frac{\partial^2 F}{\partial x \partial y}$ and $\frac{\partial^2 F}{\partial y^2}$.

Solution

$\frac{\partial F}{\partial x} = 2xy$ and $\frac{\partial F}{\partial y} = x^2 + 3y^2$.

So, $\frac{\partial^2 F}{\partial x^2} = 2y$ and $\frac{\partial^2 F}{\partial y^2} = 6y$, while $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}(x^2 + 3y^2) = 2x$ (or we may compute $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial y}(2xy) = 2x$).