Computing \( \frac{d}{dx}(\ln x) \)

We can compute the derivative of the natural logarithm from the defining equation: \( x = e^{\ln(x)} \)

Differentiating both sides, we obtain

\[
1 = \frac{d}{dx}(x) = \frac{d}{dx}(e^{\ln(x)}) = e^{\ln(x)} \frac{d}{dx}(\ln(x)) = x \frac{d}{dx}(\ln(x))
\]

Dividing both sides of this equation by \( x \), we obtain

\[
\frac{1}{x} = \frac{d}{dx}(\ln(x))
\]
Differentiating expressions involving $\ln(x)$

**Problem**

Differentiate $f(x) = x^2 \ln(x) - x \ln(x^2)$. 
First, using the power rule for the natural logarithm, we observe that

$$\ln(x^2) = 2 \ln(x)$$

so that

$$f(x) = (x^2 - 2x) \ln(x)$$

We then compute the derivative using the product rule

$$f'(x) = (2x - 2) \ln(x) + (x^2 - 2x) \frac{1}{x}$$

$$= (2x - 2) \ln(x) + x - 2$$
Another example

Example

Compute

\[ \frac{d}{dx} \left( \ln(x^3 + 3x^2 + 1) \right) \]
A solution

Using the chain rule we have

\[
\frac{d}{dx} \left( \ln(x^3 + 3x^2 + 1) \right) = \frac{1}{x^3 + 3x^2 + 1} \cdot \frac{3x^2 + 6x}{x^3 + 3x^2 + 1}
\]

\[
= \frac{3x^2 + 6x}{x^3 + 3x^2 + 1}
\]
More generally, if $f(x)$ is any differentiable function taking only positive values, then the derivative of $\ln(f(x))$ is

\[
\frac{d}{dx} (\ln(f(x))) = \frac{d}{du} (\ln(u)) \bigg|_{u=f(x)} f'(x) = \frac{1}{u} \bigg|_{u=f(x)} f'(x) = \frac{1}{f(x)} f'(x) = \frac{f'(x)}{f(x)}
\]
Definition of logarithmic derivative

The expression \( \frac{f'(x)}{f(x)} \) is called the \textit{logarithmic derivative} of \( f(x) \) and is equal to \( \frac{d}{dx}(\ln(f(x))) \) provided that \( f(x) > 0 \) always.
The function $\ln |x|$ is differentiable everywhere except at 0.

If $x > 0$ then $\ln |x| = \ln(x)$ and $\frac{d}{dx}(\ln |x|) = \frac{1}{x}$.

If $x < 0$, then $\ln |x| = \ln(-x)$ and

$$\frac{d}{dx}(\ln |x|) = \frac{d}{dx}(\ln(-x)) = \frac{1}{-x} \frac{d}{dx}(-x) = \frac{1}{x}$$

Thus, as long as $x \neq 0$, we have

$$\frac{d}{dx}(\ln |x|) = \frac{1}{x}$$
Finding extrema in an example

Example

Find the extrema of

\[ f(x) = x \ln(x) - x. \]
We compute

$$f'(x) = \ln(x) + x\frac{1}{x} - 1 = \ln(x)$$

The only solution to $f'(x) = 0$ is $x = 1$.

We compute $f''(x) = -\frac{1}{x} > 0$ for $x > 0$ so that the graph of $f$ is concave up everywhere and, in particular, $f$ has a minimum at $x = 1$. 
Graphing a logarithmic function

Example

Sketch the graph of

\[ y = 1 + \ln(x^2 - 6x + 10). \]
A solution

We note that \( x^2 - 6x + 10 \geq 1 \) for every value of \( x \) so that the function is defined at every value of \( x \).

We compute the first derivative.

\[
\frac{d}{dx}(1 + \ln(x^2 - 6x + 10)) = \frac{2x - 6}{x^2 - 6x + 10}
\]

Which is zero only for \( x = 3 \), is negative for \( x < 3 \) and is positive for \( x > 3 \). Hence, there is a minimum at \( x = 3 \).
Solution, continued

We compute the second derivative using the quotient rule:

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{2x - 6}{x^2 - 6x + 10} \right)
\]

\[
= \frac{2(x^2 - 6x + 10) - (2x - 6)(2x - 6)}{(x^2 - 6x + 10)^2}
\]

\[
= \frac{2x^2 - 12x + 20 - 4x^2 + 24x - 36}{(x^2 - 6x + 10)^2}
\]

\[
= \frac{-2x^2 + 12x - 16}{(x^2 - 6x + 10)^2}
\]

\[
= \frac{2(2 - x)(4 - x)}{(x^2 - 6x + 10)^2}
\]

which is positive for \(x < 2\) and \(x > 4\), is negative of \(2 < x < 4\), and is zero for \(x = 2\) and \(x = 4\).
Using logarithmic differentiation

Logarithmic differentiation can be used to quickly calculate derivatives of functions with simple factors.
Logarithmic differentiation in an example

Example

Compute the logarithmic derivative of
\[ f(x) = (x + 1)^9(x - 2)^8(x + 3)^7. \]
Use the result of this computation to find \( f'(x). \)
Solution

For \(x \gg 0\), we have \(f(x) > 0\) so that we can compute the natural logarithm of \(f(x)\) as

\[
\ln(f(x)) = \ln((x + 1)^9(x - 2)^8(x + 3)^7) = 9 \ln(x + 1) + 8 \ln(x - 2) + 7 \ln(x + 3)
\]
Thus, for $x \gg 0$, the logarithmic derivative of $f(x)$ is

$$\frac{f'(x)}{f(x)} = \frac{d}{dx} (\ln(f(x)))$$

$$= \frac{9}{x + 1} + \frac{8}{x - 2} + \frac{7}{x + 3}$$

Using the fact that two rational functions which agree on infinitely many arguments must be the same function, we see that this equation holds everywhere.

Multiplying both sides of the equation by $f(x)$, we see that

$$f'(x) = 9(x + 1)^8(x - 2)^8(x + 3)^7 + 8(x + 1)^9(x - 2)^7(x + 3)^7 + 7(x + 1)^9(x - 2)^8(x + 3)^6.$$
Calculating $\frac{d}{dx}(x^x)$

**Problem**

*Compute*

$$\frac{d}{dx}(x^x)$$
A solution

We write $x^x = e^{x \ln(x)}$.

\[
\frac{d}{dx}(x^x) = \frac{d}{dx}(e^{x \ln(x)})
\]
\[
= x^x \frac{d}{dx}(x \ln(x))
\]
\[
= x^x (\ln(x) + 1)
\]
Graphing exponential functions

Example

Sketch the graph of $y = f(x) = e^{-2x} - e^{-3x}$.
A solution

We compute

\[ f'(x) = -2e^{-2x} + 3e^{-3x} \]

Setting \( f'(x) = 0 \), we must solve for

\[ 2e^{-2x} = 3e^{-3x} \]

Applying the natural logarithm function, this reduces to

\[ \ln(2) - 2x = \ln(3) - 3x \]

Adding \( 3x \) to both sides of this equation and subtracting \( \ln(2) \), we find

\[ x = \ln(3/2) \]
Solution, continued

Differentiating again, we find

\[ f''(x) = 4e^{-2x} - 9e^{-3x} \]

To find the potential inflection points we set \( f''(x) = 0 \) and apply \( \ln \) obtaining

\[ \ln(4) - 2x = \ln(9) - 3x \]

Or

\[ x = \ln(9/4) = 2\ln(3/2) \]

We check that \( f''(x) < 0 \) for \( x < \ln(9/4) \) and that \( f''(x) > 0 \) for \( x > \ln(9/4) \). Hence, the graph is concave down before \( \ln(9/4) \) and concave up thereafter. In particular, there is an inflection point at \( \ln(9/4) \) and a maximum at \( \ln(3/2) \).
The graph of $y = e^{-2x} - e^{-3x}$
Example

Differentiate

\[ f(x) = \sqrt[7]{\frac{x^9 - 8x + 1}{x^{10} + 8x^2 - 4}} \]
We compute

\[
\ln(f(x)) = \ln\left(\left(\frac{x^9 - 8x + 1}{x^{10} + 8x^2 - 4}\right)^{\frac{1}{7}}\right)
\]

\[
= \frac{1}{7}(\ln(x^9 - 8x + 1) - \ln(x^{10} + 8x^2 - 4))
\]

Differentiating, we have

\[
\frac{f'}{f} = \frac{d}{dx} \ln(f(x))
\]

\[
= \frac{1}{7} \left(\frac{9x^8 - 8}{x^9 - 8x + 1} - \frac{10x^9 + 16x}{x^{10} + 8x^2 - 4}\right)
\]

Giving

\[
f'(x) = \frac{1}{7} \left(\sqrt[7]{\frac{x^9 - 8x + 1}{x^{10} + 8x^2 - 4}} \left(\frac{9x^8 - 8}{x^9 - 8x + 1} - \frac{10x^9 + 16x}{x^{10} + 8x^2 - 4}\right)\right)
\]
Simplifying before differentiating

Example

Compute the first derivative of

\[ f(x) = \frac{(e^{7x} - e^{-9x}) \sqrt{e^{5x}}}{e^{4x}} \]
Using the rules of exponents, we simplify the expression for $f(x)$.

$$f(x) = \frac{(e^{7x} - e^{-9x})\sqrt{e^{5x}}}{e^{4x}}$$

$$= (e^{7x} - e^{-9x})e^{\frac{5x}{2}}e^{-4x}$$

$$= (e^{7x} - e^{-9x})e^{\frac{-3}{2}x}$$

$$= e^{\frac{11}{2}x} - e^{\frac{-21}{2}x}$$

So,

$$f'(x) = \frac{1}{2}(11e^{\frac{11}{2}x} + 21e^{\frac{-21}{2}x})$$
Example
Sketch the graph of \( y = f(x) = \ln(x^2 + 1) \)
A solution

We compute

\[ f'(x) = \frac{2x}{x^2 + 1} \]

which is zero only at \( x = 0 \), is negative for \( x < 0 \) and is positive for \( x > 0 \). Hence, \( f \) is decreasing for negative values of \( x \), has a minimum at \((0, 0)\), and is increasing thereafter.
We compute the second derivative using the quotient rule.

\[
f''(x) = \frac{d}{dx} \left( \frac{2x}{x^2 + 1} \right)
\]

\[
= \frac{2(x^2 + 1) - 2x(2x)}{(x^2 + 1)^2}
\]

\[
= \frac{-2x^2 + 2}{x^4 + 2x^2 + 1}
\]

which is zero for \(x = \pm 1\), is negative when \(|x| < 1\), and is positive otherwise. Hence, the graph is concave up in the region where \(|x| < 1\), concave down where \(|x| > 1\) and has inflection points at \(\pm 1\).
Graph of $y = \ln(x^2 + 1)$
Example

Sketch the graph of \( y = f(x) = \ln(1 + \ln(x)^2) \)
A solution

We compute $f'(x)$ using the chain rule:

$$f'(x) = \frac{d}{dx} (\ln(1 + \ln(x)^2))$$

$$= \frac{d}{dx} (1 + \ln(x)^2)$$

$$= \frac{2 \ln(x) \frac{1}{x}}{1 + \ln(x)^2}$$

$$= \frac{2 \ln(x)}{x + x \ln(x)^2}$$

As $f$ is only defined for $x > 0$, we consider only these values. As the denominator of $f'$ is positive for all $x > 0$, we see that $f'(x) < 0$ for $0 < x < 1$, $f'(1) = 0$, and $f'(x) > 0$ for $x > 0$. Thus, $f$ is decreasing in the range $0 < x < 1$, has a minimum at $(1,0)$, and is increasing thereafter.
Solution, continued

We compute the second derivative using the quotient rule.

\[
\frac{d}{dx} \left( \frac{2 \ln(x)}{x + x \ln(x)^2} \right) = \frac{2 \ln(x) \left( 1 + \ln(x)^2 \right) - 2 \ln(x) \left( 1 + \ln(x)^2 + x \ln(x)^{\frac{1}{x}} \right)}{x^2 (1 + \ln(x)^2)^2} = \frac{2 + 2 \ln(x)^2 - 2 \ln(x) \left( 1 + \ln(x)^2 + 2 \ln(x) \right)}{x^2 (1 + \ln(x)^2)^2} = \frac{-2(\ln(x)^3 + \ln(2)^2 + \ln(x) - 1)}{x^2 (1 + \ln(x)^2)^2}
\]
Solving for the inflection points

The derivative of \( f''(x) \) is shown. The denominator of \( f''(x) \) is always positive while the numerator has the form \(-\frac{2}{G(ln(x))}\) where \( G(Y) = Y^3 + Y^2 + Y - 1\). 

\( G'(Y) = 3Y^2 + 2Y + 1 \) is always positive as its discriminant, \(-12\), is negative. The only root \( \alpha \) to \( G(Y) = 0 \) occurs at \( \approx 0.545 \). Hence, the only root to \( f''(x) = 0 \) occurs at \( e^\alpha \approx e^{0.545} \approx 1.725 \).
Graph of $y = \ln(1 + \ln(x)^2)$