Computing $\frac{d}{dx}(\ln x)$

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$= e^{\ln(x)} \frac{d}{dx}(\ln(x))$
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$$= e^{\ln(x)} \frac{d}{dx}(\ln(x))$$

$$= x \frac{d}{dx}(\ln(x))$$
Computing $\frac{d}{dx}(\ln x)$

We can compute the derivative of the natural logarithm from the defining equation: $x = e^{\ln(x)}$

Differentiating both sides, we obtain

\[
1 = \frac{d}{dx}(x) = \frac{d}{dx}(e^{\ln(x)}) = e^{\ln(x)} \frac{d}{dx}(\ln(x)) = x \frac{d}{dx}(\ln(x))
\]

Dividing both sides of this equation by $x$, we obtain

\[
\frac{1}{x} = \frac{d}{dx}(\ln(x))
\]
Problem

Differentiate \( f(x) = x^2 \ln(x) - x \ln(x^2) \).
A solution

First, using the power rule for the natural logarithm, we observe that
A solution

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\[ \ln(x^2) = 2\ln(x) \]
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We then compute the derivative using the product rule.
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$$\ln(x^2) = 2 \ln(x)$$

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We then compute the derivative using the product rule

$$f'(x) = (2x - 2) \ln(x) + (x^2 - 2x) \frac{1}{x}$$
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\[ \ln(x^2) = 2 \ln(x) \]

so that

\[ f(x) = (x^2 - 2x) \ln(x) \]

We then compute the derivative using the product rule

\[
\begin{align*}
  f'(x) &= (2x - 2) \ln(x) + (x^2 - 2x) \frac{1}{x} \\
  &= (2x - 2) \ln(x) + x - 2
\end{align*}
\]
Another example

Example

Compute

\[ \frac{d}{dx} \left( \ln(x^3 + 3x^2 + 1) \right) \]
A solution

Using the chain rule we have
Using the chain rule we have

\[
\frac{d}{dx} \left( \ln(x^3 + 3x^2 + 1) \right) = \frac{1}{x^3 + 3x^2 + 1} (3x^2 + 6x)
\]
Using the chain rule we have

\[ \frac{d}{dx}(\ln(x^3 + 3x^2 + 1)) = \frac{1}{x^3 + 3x^2 + 1} \frac{3x^2 + 6x}{3x^2 + 6x} = \frac{3x^2 + 6x}{x^3 + 3x^2 + 1} \]
Logarithmic derivatives

More generally, if \( f(x) \) is any differentiable function taking only positive values, then the derivative of \( \ln(f(x)) \) is
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More generally, if \( f(x) \) is any differentiable function taking only positive values, then the derivative of \( \ln(f(x)) \) is

\[
\frac{d}{dx} (\ln(f(x))) = \left. \frac{d}{du} (\ln(u)) \right|_{u=f(x)} f'(x)
\]
More generally, if $f(x)$ is any differentiable function taking only positive values, then the derivative of $\ln(f(x))$ is

$$\frac{d}{dx} (\ln(f(x))) = \frac{d}{du} (\ln(u)) \bigg|_{u=f(x)} f'(x)$$

$$= \frac{1}{u} \bigg|_{u=f(x)} f'(x)$$
Logarithmic derivatives

More generally, if $f(x)$ is any differentiable function taking only positive values, then the derivative of $\ln(f(x))$ is

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\frac{d}{dx}(\ln(f(x))) = \left. \frac{d}{du}(\ln(u)) \right|_{u=f(x)} f'(x)
$$

$$
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$$

$$
= \frac{1}{f(x)} f'(x)
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More generally, if $f(x)$ is any differentiable function taking only positive values, then the derivative of $\ln(f(x))$ is

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\]

\[
= \frac{1}{u} \bigg|_{u=f(x)} f'(x)
\]

\[
= \frac{1}{f(x)} f'(x)
\]

\[
= \frac{f'(x)}{f(x)}
\]
Definition of logarithmic derivative

The expression \( \frac{f'(x)}{f(x)} \) is called the logarithmic derivative of \( f(x) \) and is equal to \( \frac{d}{dx} (\ln(f(x))) \) provided that \( f(x) > 0 \) always.
The function $\ln|x|$ is differentiable everywhere except at 0.
The function \( \ln |x| \) is differentiable everywhere except at 0.

If \( x > 0 \) then \( \ln |x| = \ln(x) \) and \( \frac{d}{dx}(\ln |x|) = \frac{1}{x} \).
Derivative of $\ln |x|$ 

The function $\ln |x|$ is differentiable everywhere except at 0.

If $x > 0$ then $\ln |x| = \ln(x)$ and \[
\frac{d}{dx} (\ln |x|) = \frac{1}{x}.
\]

If $x < 0$, then $\ln |x| = \ln(-x)$ and 
\[
\frac{d}{dx} (\ln |x|) = \frac{d}{dx} (\ln(-x)) = \frac{1}{-x} \frac{d}{dx} (-x) = \frac{1}{x}.
\]
The function $\ln |x|$ is differentiable everywhere except at 0. If $x > 0$ then $\ln |x| = \ln(x)$ and $\frac{d}{dx}(\ln |x|) = \frac{1}{x}$.

If $x < 0$, then $\ln |x| = \ln(-x)$ and

$$\frac{d}{dx}(\ln |x|) = \frac{d}{dx}(\ln(-x)) = \frac{1}{-x} \frac{d}{dx}(-x) = \frac{1}{x}.$$ 

Thus, as long as $x \neq 0$, we have

$$\frac{d}{dx}(\ln |x|) = \frac{1}{x}.$$
Finding extrema in an example

**Example**

Find the extrema of

\[ f(x) = x \ln(x) - x. \]
We compute

\[ f'(x) = \ln(x) + x \frac{1}{x} - 1 = \ln(x) \]
A solution

We compute

\[ f'(x) = \ln(x) + x \cdot \frac{1}{x} - 1 = \ln(x) \]

The only solution to \( f'(x) = 0 \) is \( x = 1 \).
We compute

\[ f'(x) = \ln(x) + \frac{1}{x} - 1 = \ln(x) \]

The only solution to \( f'(x) = 0 \) is \( x = 1 \).

We compute \( f''(x) = \frac{1}{x} > 0 \) for \( x > 0 \) so that the graph of \( f \) is concave up everywhere and, in particular, \( f \) has a minimum at \( x = 1 \).
Graphing a logarithmic function

Example
Sketch the graph of
\[ y = 1 + \ln(x^2 - 6x + 10). \]
We note that $x^2 - 6x + 10 \geq 1$ for every value of $x$ so that the function is defined at every value of $x$. 
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We note that $x^2 - 6x + 10 \geq 1$ for every value of $x$ so that the function is defined at every value of $x$.

We compute the first derivative.

$$\frac{d}{dx}(1 + \ln(x^2 - 6x + 10)) = \frac{2x - 6}{x^2 - 6x + 10}$$
We note that $x^2 - 6x + 10 \geq 1$ for every value of $x$ so that the function is defined at every value of $x$.

We compute the first derivative.

$$
\frac{d}{dx} \left( 1 + \ln(x^2 - 6x + 10) \right) = \frac{2x - 6}{x^2 - 6x + 10}
$$

Which is zero only for $x = 3$, is negative for $x < 3$ and is positive for $x > 3$. 
We note that $x^2 - 6x + 10 \geq 1$ for every value of $x$ so that the function is defined at every value of $x$.

We compute the first derivative.

$$\frac{d}{dx}(1 + \ln(x^2 - 6x + 10)) = \frac{2x - 6}{x^2 - 6x + 10}$$

Which is zero only for $x = 3$, is negative for $x < 3$ and is positive for $x > 3$. Hence, there is a minimum at $x = 3$. 
We compute the second derivative using the quotient rule:
Solution, continued

We compute the second derivative using the quotient rule:

\[
\frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{2x - 6}{x^2 - 6x + 10} \right)
\]
Solution, continued

We compute the second derivative using the quotient rule:

\[
\begin{align*}
\frac{d^2 y}{dx^2} &= \frac{d}{dx} \left( \frac{2x - 6}{x^2 - 6x + 10} \right) \\
&= \frac{2(x^2 - 6x + 10) - (2x - 6)(2x - 6)}{(x^2 - 6x + 10)^2}
\end{align*}
\]
Solution, continued

We compute the second derivative using the quotient rule:

\[
\frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{2x - 6}{x^2 - 6x + 10} \right)
\]

\[
= \frac{2(x^2 - 6x + 10) - (2x - 6)(2x - 6)}{(x^2 - 6x + 10)^2}
\]

\[
= \frac{2x^2 - 12x + 20 - 4x^2 + 24x - 36}{(x^2 - 6x + 10)^2}
\]
We compute the second derivative using the quotient rule:

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= \frac{2x^2 - 12x + 20 - 4x^2 + 24x - 36}{(x^2 - 6x + 10)^2}
\]

\[
= \frac{-2x^2 + 12x - 16}{(x^2 - 6x + 10)^2}
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We compute the second derivative using the quotient rule:

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\frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{2x - 6}{x^2 - 6x + 10} \right)
= \frac{2(x^2 - 6x + 10) - (2x - 6)(2x - 6)}{(x^2 - 6x + 10)^2}
= \frac{2x^2 - 12x + 20 - 4x^2 + 24x - 36}{(x^2 - 6x + 10)^2}
= \frac{-2x^2 + 12x - 16}{(x^2 - 6x + 10)^2}
= \frac{2(2 - x)(4 - x)}{(x^2 - 6x + 10)^2}
\]
Solution, continued

We compute the second derivative using the quotient rule:

\[
\frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{2x - 6}{x^2 - 6x + 10} \right)
\]

\[
= \frac{2(x^2 - 6x + 10) - (2x - 6)(2x - 6)}{(x^2 - 6x + 10)^2}
\]

\[
= \frac{2x^2 - 12x + 20 - 4x^2 + 24x - 36}{(x^2 - 6x + 10)^2}
\]

\[
= \frac{-2x^2 + 12x - 16}{(x^2 - 6x + 10)^2}
\]

\[
= \frac{2(2 - x)(4 - x)}{(x^2 - 6x + 10)^2}
\]

which is positive for \(x < 2\) and \(x > 4\), is negative of \(2 < x < 4\), and is zero for \(x = 2\) and \(x = 4\).
Logarithmic differentiation can be used to quickly calculate derivatives of functions with simple factors.
Logarithmic differentiation in an example

**Example**

Compute the logarithmic derivative of $f(x) = (x + 1)^9(x - 2)^8(x + 3)^7$. Use the result of this computation to find $f'(x)$. 
Solution

For $x \gg 0$, we have $f(x) > 0$ so that we can compute the natural logarithm of $f(x)$ as
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$$\ln(f(x)) =$$
Solution

For $x \gg 0$, we have $f(x) > 0$ so that we can compute the natural logarithm of $f(x)$ as

$$\ln(f(x)) = \ln((x + 1)^9(x - 2)^8(x + 3)^7) =$$
For $x \gg 0$, we have $f(x) > 0$ so that we can compute the natural logarithm of $f(x)$ as

$$\ln(f(x)) = \ln((x + 1)^9(x - 2)^8(x + 3)^7) = \ln((x + 1)^9) + \ln((x - 2)^8) + \ln((x + 3)^7)$$
Solution

For $x \gg 0$, we have $f(x) > 0$ so that we can compute the natural logarithm of $f(x)$ as

$$
\ln(f(x)) = \\
\ln((x + 1)^9(x - 2)^8(x + 3)^7) = \\
\ln((x + 1)^9) + \ln((x - 2)^8) + \ln((x + 3)^7) = \\
9\ln(x + 1) + 8\ln(x - 2) + 7\ln(x + 3)
$$
Thus, for $x \gg 0$, the logarithmic derivative of $f(x)$ is
Thus, for $x \gg 0$, the logarithmic derivative of $f(x)$ is

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}(\ln(f(x)))$$
Thus, for $x \gg 0$, the logarithmic derivative of $f(x)$ is

$$\frac{f'(x)}{f(x)} = \frac{d}{dx} \ln(f(x))$$

$$= \frac{9}{x + 1} + \frac{8}{x - 2} + \frac{7}{x + 3}$$
Thus, for \( x \gg 0 \), the logarithmic derivative of \( f(x) \) is

\[
\frac{f'(x)}{f(x)} = \frac{d}{dx} (\ln(f(x)))
\]

\[
= \frac{9}{x + 1} + \frac{8}{x - 2} + \frac{7}{x + 3}
\]

Using the fact that two rational functions which agree on infinitely many arguments must be the same function, we see that this equation holds everywhere.
Thus, for $x \gg 0$, the logarithmic derivative of $f(x)$ is

$$\frac{f'(x)}{f(x)} = \frac{d}{dx} \left( \ln(f(x)) \right)$$

$$= \frac{9}{x+1} + \frac{8}{x-2} + \frac{7}{x+3}$$

Using the fact that two rational functions which agree on infinitely many arguments must be the same function, we see that this equation holds everywhere.

Multiplying both sides of the equation by $f(x)$, we see that $f'(x) = 9(x+1)^8(x-2)^8(x+3)^7 + 8(x+1)^9(x-2)^7(x+3)^7 + 7(x+1)^9(x-2)^8(x+3)^6$. 
Calculating \( \frac{d}{dx}(x^x) \)

**Problem**

*Compute*

\[
\frac{d}{dx}(x^x)
\]
A solution

We write \( x^x = e^{x \ln(x)} \).
A solution

We write $x^x = e^{x \ln(x)}$.

\[
\frac{d}{dx}(x^x) = \frac{d}{dx}(e^{x \ln(x)})
\]
We write \( x^x = e^{x \ln(x)} \).

\[
\frac{d}{dx}(x^x) = \frac{d}{dx}(e^{x \ln(x)})
= x^x \frac{d}{dx}(x \ln(x))
\]
A solution

We write $x^x = e^{x \ln(x)}$.

\[
\frac{d}{dx}(x^x) = \frac{d}{dx}(e^{x \ln(x)}) \\
= x^x \frac{d}{dx}(x \ln(x)) \\
= x^x (\ln(x) + 1)
\]
Graphing exponential functions

Example

Sketch the graph of $y = f(x) = e^{-2x} - e^{-3x}$. 
A solution

We compute

\[ f'(x) = -2e^{-2x} + 3e^{-3x} \]
A solution

We compute

\[ f'(x) = -2e^{-2x} + 3e^{-3x} \]

Setting \( f'(x) = 0 \), we must solve for

\[ 2e^{-2x} = 3e^{-3x} \]
We compute
\[ f'(x) = -2e^{-2x} + 3e^{-3x} \]

Setting \( f'(x) = 0 \), we must solve for
\[ 2e^{-2x} = 3e^{-3x} \]

Applying the natural logarithm function, this reduces to
\[ \ln(2) - 2x = \ln(3) - 3x \]
A solution

We compute

\[ f'(x) = -2e^{-2x} + 3e^{-3x} \]

Setting \( f'(x) = 0 \), we must solve for

\[ 2e^{-2x} = 3e^{-3x} \]

Applying the natural logarithm function, this reduces to

\[ \ln(2) - 2x = \ln(3) - 3x \]

Adding \( 3x \) to both sides of this equation and subtracting \( \ln(2) \), we find

\[ x = \ln(3/2) \]
Solution, continued

Differentiating again, we find

\[ f''(x) = 4e^{-2x} - 9e^{-3x} \]
Differentiating again, we find

\[ f''(x) = 4e^{-2x} - 9e^{-3x} \]

To find the potential inflection points we set \( f''(x) = 0 \) and apply \( \ln \) obtaining

\[ \ln(4) - 2x = \ln(9) - 3x \]
Differentiating again, we find

\[ f''(x) = 4e^{-2x} - 9e^{-3x} \]

To find the potential inflection points we set \( f''(x) = 0 \) and apply \( \ln \) obtaining

\[ \ln(4) - 2x = \ln(9) - 3x \]

Or

\[ x = \ln(9/4) = 2\ln(3/2) \]
Solution, continued

Differentiating again, we find

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To find the potential inflection points we set \( f''(x) = 0 \) and apply \ln \) obtaining

\[ \ln(4) - 2x = \ln(9) - 3x \]

Or

\[ x = \ln(9/4) = 2\ln(3/2) \]

We check that \( f''(x) < 0 \) for \( x < \ln(9/4) \) and that \( f''(x) > 0 \) for \( x > \ln(9/4) \). Hence, the graph is concave down before \( \ln(9/4) \) and concave up thereafter. In particular, there is an inflection point at \( \ln(9/4) \) and a maximum at \( \ln(3/2) \).
The graph of $y = e^{-2x} - e^{-3x}$
Using logarithmic differentiation

Example

Differentiate

\[ f(x) = \sqrt[7]{\frac{x^9 - 8x + 1}{x^{10} + 8x^2 - 4}} \]
A solution

We compute
We compute

\[ \ln(f(x)) = \ln\left(\frac{x^9 - 8x + 1}{x^{10} + 8x^2 - 4}^{\frac{1}{7}}\right) \]
A solution

We compute

\[
\ln(f(x)) = \ln\left(\frac{x^9 - 8x + 1}{x^{10} + 8x^2 - 4}\right)^{\frac{1}{7}}
\]

\[
= \frac{1}{7}(\ln(x^9 - 8x + 1) - \ln(x^{10} + 8x^2 - 4))
\]
A solution

We compute

\[
\ln(f(x)) = \ln\left(\left(\frac{x^9 - 8x + 1}{x^{10} + 8x^2 - 4}\right)^{\frac{1}{7}}\right) \\
= \frac{1}{7}(\ln(x^9 - 8x + 1) - \ln(x^{10} + 8x^2 - 4))
\]

Differentiating, we have
A solution

We compute

\[
\ln(f(x)) = \ln\left(\left(\frac{x^9 - 8x + 1}{x^{10} + 8x^2 - 4}\right)^{\frac{1}{7}}\right)
\]

\[
= \frac{1}{7}(\ln(x^9 - 8x + 1) - \ln(x^{10} + 8x^2 - 4))
\]

Differentiating, we have

\[
\frac{f'}{f} = \frac{d}{dx} \ln(f(x))
\]
A solution

We compute

\[
\ln(f(x)) = \ln\left(\left(\frac{x^9 - 8x + 1}{x^{10} + 8x^2 - 4}\right)^{\frac{1}{7}}\right)
\]

\[
= \frac{1}{7}(\ln(x^9 - 8x + 1) - \ln(x^{10} + 8x^2 - 4))
\]

Differentiating, we have

\[
\frac{f'}{f} = \frac{d}{dx} \ln(f(x))
\]

\[
= \frac{1}{7} \cdot \frac{9x^8 - 8}{x^9 - 8x + 1} - \frac{1}{7} \cdot \frac{10x^9 + 16x}{x^{10} + 8x^2 - 4}
\]
A solution

We compute

\[
\ln(f(x)) = \ln\left(\frac{x^9 - 8x + 1}{x^{10} + 8x^2 - 4}\right)^{\frac{1}{7}}
\]

\[
= \frac{1}{7}(\ln(x^9 - 8x + 1) - \ln(x^{10} + 8x^2 - 4))
\]

Differentiating, we have

\[
\frac{f'}{f} = \frac{d}{dx} \ln(f(x))
\]

\[
= \frac{1}{7} \frac{9x^8 - 8}{x^9 - 8x + 1} - \frac{1}{7} \frac{10x^9 + 16x}{x^{10} + 8x^2 - 4}
\]

Giving

\[
f'(x) = \frac{1}{7} \left(\sqrt[7]{\frac{x^9 - 8x + 1}{x^{10} + 8x^2 - 4}}\right)\left(\frac{9x^8 - 8}{x^9 - 8x + 1} - \frac{10x^9 + 16x}{x^{10} + 8x^2 - 4}\right)
\]
Simplifying before differentiating

Example

Compute the first derivative of

\[ f(x) = \frac{(e^{7x} - e^{-9x}) \sqrt{e^{5x}}}{e^{4x}} \]
A solution

Using the rules of exponents, we simplify the expression for $f(x)$. 
Using the rules of exponents, we simplify the expression for \( f(x) \).

\[
f(x) = \frac{(e^{7x} - e^{-9x})\sqrt{e^{5x}}}{e^{4x}}
\]
Using the rules of exponents, we simplify the expression for $f(x)$.

$$f(x) = \frac{(e^{7x} - e^{-9x})\sqrt{e^{5x}}}{e^{4x}}$$

$$= (e^{7x} - e^{-9x})e^{\frac{5}{2}x}e^{-4x}$$
A solution

Using the rules of exponents, we simplify the expression for $f(x)$.

\[
f(x) = \frac{(e^{7x} - e^{-9x}) \sqrt{e^{5x}}}{e^{4x}} \\
= (e^{7x} - e^{-9x})e^{\frac{5}{2}x}e^{-4x} \\
= (e^{7x} - e^{-9x})e^{\frac{-3}{2}x}
\]
A solution

Using the rules of exponents, we simplify the expression for \( f(x) \).

\[
f(x) = \frac{(e^{7x} - e^{-9x})\sqrt{e^{5x}}}{e^{4x}}
\]

\[
= (e^{7x} - e^{-9x})e^{\frac{5}{2}x}e^{-4x}
\]

\[
= (e^{7x} - e^{-9x})e^{\frac{-3}{2}x}
\]

\[
= e^{\frac{11}{2}x} - e^{\frac{-21}{2}x}
\]
Using the rules of exponents, we simplify the expression for \( f(x) \).

\[ f(x) = \frac{(e^{7x} - e^{-9x})\sqrt{e^{5x}}}{e^{4x}} \]
\[ = (e^{7x} - e^{-9x})e^{\frac{5}{2}x}e^{-4x} \]
\[ = (e^{7x} - e^{-9x})e^{\frac{-3}{2}x} \]
\[ = e^{\frac{11}{2}x} - e^{\frac{-21}{2}x} \]

So,

\[ f'(x) = \frac{1}{2}(11e^{\frac{11}{2}x} + 21e^{\frac{-21}{2}x}) \]
Example

Sketch the graph of $y = f(x) = \ln(x^2 + 1)$
A solution

We compute

\[ f'(x) = \frac{2x}{x^2 + 1} \]
A solution

We compute

\[ f'(x) = \frac{2x}{x^2 + 1} \]

which is zero only at \( x = 0 \), is negative for \( x < 0 \) and is positive for \( x > 0 \).
A solution

We compute

\[ f'(x) = \frac{2x}{x^2 + 1} \]

which is zero only at \( x = 0 \), is negative for \( x < 0 \) and is positive for \( x > 0 \). Hence, \( f \) is decreasing for negative values of \( x \), has a minimum at \((0, 0)\), and is increasing thereafter.
Solution, continued

We compute the second derivative using the quotient rule.
We compute the second derivative using the quotient rule.

\[ f''(x) = \frac{d}{dx} \left( \frac{2x}{x^2 + 1} \right) \]
Solution, continued

We compute the second derivative using the quotient rule.

\[ f''(x) = \frac{d}{dx} \left( \frac{2x}{x^2 + 1} \right) \]
\[ = \frac{2(x^2 + 1) - 2x(2x)}{(x^2 + 1)^2} \]
We compute the second derivative using the quotient rule.

\[
f''(x) = \frac{d}{dx} \left( \frac{2x}{x^2 + 1} \right) = \frac{2(x^2 + 1) - 2x(2x)}{(x^2 + 1)^2} = \frac{-2x^2 + 2}{x^4 + 2x^2 + 1}
\]
We compute the second derivative using the quotient rule.

\[
\begin{align*}
    f''(x) &= \frac{d}{dx} \left( \frac{2x}{x^2 + 1} \right) \\
    &= \frac{2(x^2 + 1) - 2x(2x)}{(x^2 + 1)^2} \\
    &= \frac{-2x^2 + 2}{x^4 + 2x^2 + 1}
\end{align*}
\]

which is zero for \(x = \pm 1\), is negative when \(|x| < 1\), and is positive otherwise.
We compute the second derivative using the quotient rule.

\[
\frac{d}{dx} \left( \frac{2x}{x^2 + 1} \right) = \frac{2(x^2 + 1) - 2x(2x)}{(x^2 + 1)^2} = \frac{-2x^2 + 2}{x^4 + 2x^2 + 1}
\]

which is zero for \( x = \pm 1 \), is negative when \(|x| < 1\), and is positive otherwise. Hence, the graph is concave up in the region where \(|x| < 1\), concave down where \(|x| > 1\) and has inflection points at \( \pm 1 \).
Graph of \( y = \ln(x^2 + 1) \)
Example

Sketch the graph of $y = f(x) = \ln(1 + \ln(x)^2)$
A solution

We compute $f'(x)$ using the chain rule:
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$$f'(x) = \frac{d}{dx}(\ln(1 + \ln(x)^2))$$
A solution

We compute $f'(x)$ using the chain rule:

$$f'(x) = \frac{d}{dx} (\ln(1 + \ln(x)^2))$$

$$= \frac{d}{dx} (1 + \ln(x)^2) \frac{1}{1 + \ln(x)^2}$$
We compute $f'(x)$ using the chain rule:

$$f'(x) = \frac{d}{dx}(\ln(1 + \ln(x)^2))$$

$$= \frac{d}{dx}(1 + \ln(x)^2)$$

$$= \frac{1}{1 + \ln(x)^2} \cdot 2 \ln(x) \frac{1}{x}$$

$$= \frac{2 \ln(x)}{1 + \ln(x)^2}$$
A solution

We compute $f'(x)$ using the chain rule:

\[
  f'(x) = \frac{d}{dx} \left( \ln(1 + \ln(x)^2) \right)
\]

\[
  = \frac{d}{dx} \left( 1 + \ln(x)^2 \right) \cdot \frac{1}{1 + \ln(x)^2}
\]

\[
  = \frac{2 \ln(x) \frac{1}{x}}{1 + \ln(x)^2}
\]

\[
  = \frac{2 \ln(x)}{x + x \ln(x)^2}
\]
A solution

We compute \( f'(x) \) using the chain rule:

\[
f'(x) = \frac{d}{dx} (\ln(1 + \ln(x)^2))
\]
\[
= \frac{d}{dx} (1 + \ln(x)^2) \quad \frac{1}{1 + \ln(x)^2}
\]
\[
= \frac{2 \ln(x) \frac{1}{x}}{1 + \ln(x)^2}
\]
\[
= \frac{2 \ln(x)}{x + x \ln(x)^2}
\]

As \( f \) is only defined for \( x > 0 \), we consider only these values.
A solution

We compute $f'(x)$ using the chain rule:

$$f'(x) = \frac{d}{dx} (\ln(1 + \ln(x)^2))$$

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$$= \frac{2 \ln(x)}{x + x \ln(x)^2}$$

As $f$ is only defined for $x > 0$, we consider only these values. As the denominator of $f'$ is positive for all $x > 0$, we see that $f'(x) < 0$ for $0 < x < 1$, $f'(1) = 0$, and $f'(x) > 0$ for $x > 0$. 

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Math 16A (Autumn 2005)
A solution

We compute $f'(x)$ using the chain rule:

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$$= \frac{d}{dx} (1 + \ln(x)^2)$$

$$= \frac{2 \ln(x) \frac{1}{x}}{1 + \ln(x)^2}$$

$$= \frac{2 \ln(x)}{x + x \ln(x)^2}$$

As $f$ is only defined for $x > 0$, we consider only these values. As the denominator of $f'$ is positive for all $x > 0$, we see that $f'(x) < 0$ for $0 < x < 1$, $f'(1) = 0$, and $f'(x) > 0$ for $x > 0$. Thus, $f$ is decreasing in the range $0 < x < 1$, has a minimum at $(1, 0)$, and is increasing thereafter.
Solution, continued

We compute the second derivative using the quotient rule.
Solution, continued

We compute the second derivative using the quotient rule.

\[
f''(x) = \frac{d}{dx} \left( \frac{2 \ln(x)}{x + x \ln(x)^2} \right)
\]
We compute the second derivative using the quotient rule.

\[
\begin{align*}
    f''(x) &= \frac{d}{dx} \left( \frac{2 \ln(x)}{x + x \ln(x)^2} \right) \\
    &= \frac{2 \frac{1}{x} x (1 + \ln(x)^2) - 2 \ln(x)(1 + \ln(x)^2 + x2 \ln(x) \frac{1}{x})}{x^2 (1 + \ln(x)^2)^2}
\end{align*}
\]
Solution, continued

We compute the second derivative using the quotient rule.

\[ f''(x) = \frac{d}{dx} \left( \frac{2 \ln(x)}{x + x \ln(x)^2} \right) \]

\[ = \frac{2 \frac{1}{x} x (1 + \ln(x)^2) - 2 \ln(x) (1 + \ln(x)^2 + x 2 \ln(x) \frac{1}{x})}{x^2 (1 + \ln(x)^2)^2} \]

\[ = \frac{2 + 2 \ln(x)^2 - 2 \ln(x) (1 + \ln(x)^2 + 2 \ln(x))}{x^2 (1 + \ln(x)^2)^2} \]
Solution, continued

We compute the second derivative using the quotient rule.

\[
\begin{align*}
f''(x) &= \frac{d}{dx} \left( \frac{2 \ln(x)}{x + x \ln(x)^2} \right) \\
    &= \frac{\frac{2}{x}(1 + \ln(x)^2) - 2 \ln(x)(1 + \ln(x)^2 + x2 \ln(x)\frac{1}{x})}{x^2(1 + \ln(x)^2)^2} \\
    &= \frac{2 + 2 \ln(x)^2 - 2 \ln(x)(1 + \ln(x)^2 + 2 \ln(x))}{x^2(1 + \ln(x)^2)^2} \\
    &= \frac{-2(\ln(x)^3 + \ln(2)^2 + \ln(x) - 1)}{x^2(1 + \ln(x)^2)^2}
\end{align*}
\]
Solving for the inflection points

The denominator of $f''(x)$ is always positive while the numerator have the form $-2G(\ln(x))$ where $G(Y) = Y^3 + Y^2 + Y - 1$. 
Solving for the inflection points

The denominator of $f''(x)$ is always positive while the numerator have the form $-2G(ln(x))$ where $G(Y) = Y^3 + Y^2 + Y - 1$. $G'(Y) = 3Y^2 + 2Y + 1$ is always positive as its discriminant, $-12$, is negative.
Solving for the inflection points

The denominator of $f''(x)$ is always positive while the numerator have the form $-2G(\ln(x))$ where $G(Y) = Y^3 + Y^2 + Y - 1$. $G'(Y) = 3Y^2 + 2Y + 1$ is always positive as its discriminant, $-12$, is negative. The only root $\alpha$ to $G(Y) = 0$ occurs at $\approx 0.545$. 
Solving for the inflection points

The denominator of $f''(x)$ is always positive while the numerator have the form $-2G(\ln(x))$ where $G(Y) = Y^3 + Y^2 + Y - 1$. $G'(Y) = 3Y^2 + 2Y + 1$ is always positive as its discriminant, $-12$, is negative.

The only root $\alpha$ to $G(Y) = 0$ occurs at $\approx 0.545$.

Hence, the only root to $f''(x) = 0$ occurs at $e^{\alpha} \approx e^{0.545} \approx 1.725$. 

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Graph of $y = \ln(1 + \ln(x)^2)$