In sketching the graph of a function, one should look for basic qualitative features.

- Relative Extrema
- Inflection Points
- Intercepts
- Concavity
- Asymptotes
Graphing a cubic

Example

Graph \( f(x) = x^3 - 2x^2 + x - 1 \) for \(-3 \leq x \leq 3\).
To find the relative extrema we look at the endpoints of the domain of the function (if any) and at the zeros of the derivative.

At the endpoints

- $f(-3) = -27 - 18 - 3 - 1 = -49$
- $f(3) = 27 - 18 + 3 - 1 = 11$. 

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We compute the first derivative

\[ f'(x) = 3x^2 - 4x + 1 \]
\[ = (3x - 1)(x - 1) \]

So that \( f'(x) = 0 \) for \( x = \frac{1}{3} \)
and \( x = 1 \).

We evaluate the function at these points obtaining
\[ f\left(\frac{1}{3}\right) = \frac{1}{27} - \frac{6}{27} + \frac{9}{27} - \frac{27}{27} = -\frac{23}{27} \]
and
\[ f(1) = 1 - 2 + 1 - 1 = -1. \]
Finding inflection points and determining concavity

Concavity may change where \( f''(x) = 0 \). In our case, we have \( f''(x) = 6x - 4 \) so that the only possible inflection point occurs at \( x = \frac{2}{3} \).

As \( 6x - 4 < 0 \) when \( x < \frac{2}{3} \) and \( 6x - 4 > 0 \) when \( x > \frac{2}{3} \), we see that the graph of \( f(x) \) is concave down for \( x < \frac{2}{3} \) and is concave up when \( x > \frac{2}{3} \) and that \( x = \frac{2}{3} \) is an inflection point.
Finding the intercepts

To find the $y$-intercept we evaluate at zero:

$$f(0) = -1$$

To find the $x$-intercept, we must solve $f(x) = 0$. In our case, there is no simple algebraic solution to this problem. However, we can approximate the location of the $x$-intercept by evaluating $f$ in the range $1 \leq x \leq 2$. 
Finding the asymptotes

To find the asymptotes, we look at undefined points for the function and at \( \lim_{x \to \pm \infty} f'(x) \).

In our case, the function is continuous at every real number. So, there are no vertical asymptotes.

As \( f(x) \) is defined only in the region \(-3 \leq x \leq 3\), we need not consider asymptotes at \( \infty \).
Graphing a rational function

Example

Graph \( f(x) = x^2 - 2x + \frac{1}{x-1} \)
for \(-3 \leq x \leq 3\).
To find the relative extrema we look at the endpoints of the domain of the function (if any) and at the zeros of the derivative.

At the endpoints

- \( f(-3) = (-3)^2 - 2(-3) + 1/(-3 - 1) = 14.75 \) and

- \( f(3) = 3^2 - 2(3) + 1/(3 - 1) = 3.5. \)
We compute the first derivative

\[ f'(x) = 2x - 2 - (x - 1)^{-2} \]

\[ = \frac{2(x - 1)^3 - 1}{(x - 1)^2} \]

So that \( f'(x) = 0 \) for \( x = 1 + \sqrt[3]{\frac{1}{2}} \).

We evaluate the function at this point obtaining \( f(1 + \sqrt[3]{\frac{1}{2}}) \approx 0.889881575 \)
Finding inflection points and determining concavity

Concavity may change where \( f''(x) = 0 \).
In our case, we have \( f''(x) = 2 + 2(x - 1)^{-3} \) so that the only possible inflection point occurs at \( x = 0 \).
As \( f''(x) < 0 \) when \( 0 < x < 1 \) and \( f''(x) > 0 \) when \( x > 1 \) or \( x < 0 \), we see that the graph of \( f(x) \) is concave down for \( 0 < x < 1 \) and is concave up when \( x < 0 \) or \( x > 1 \) and that \( x = 0 \) is an inflection point.
Finding the intercepts

To find the $y$-intercept we evaluate at zero:

$$f(0) = -1$$

To find the $x$-intercept, we must solve $f(x) = 0$. In our case, there is no simple algebraic solution to this problem. However, we can approximate the location of the $x$-intercept by evaluating $f$ in the range $-0.4 \leq x \leq -0.3$. 
Finding the asymptotes

To find the asymptotes, we look at undefined points for the function and at \( \lim_{x \to \pm \infty} f'(x) \).

In our case, the function is undefined at \( x = 1 \) and there is clearly a vertical asymptote there.

As the function is only defined for \( -3 \leq x \leq 3 \), we need not consider asymptotes at infinity.
Graphing a function with fractional powers

Example

Graph \( f(x) = x + \sqrt{4 - 3x} \) for \( 0 \leq x \leq \frac{4}{3} \).
To find the relative extrema we look at the endpoints of the domain of the function (if any) and at the zeros of the derivative.

At the endpoints

- $f(0) = 2$ and
- $f\left(\frac{4}{3}\right) = 0$. 
We compute the first derivative

\[ f'(x) = 1 - \frac{3}{2}(4 - 3x)^{-\frac{1}{2}} \]

So that \( f'(x) = 0 \) for \( x = \frac{7}{12} \).

We evaluate the function at this point obtaining \( f\left(\frac{7}{12}\right) = \frac{25}{12} \).
Concavity may change where $f''(x) = 0$. In our case, we have $f''(x) = \frac{-9}{4}(4 - 3x)^{-\frac{3}{2}}$. In the range where $f''(x)$ is defined, $f''(x)$ is always negative so that the graph of $f(x)$ is always concave down. Note, however, that $f''(x)$ tends to zero as $x$ approaches $\frac{4}{3}$. 
Finding the intercepts

To find the \( y \)-intercept we evaluate at zero: 
\[ f(0) = 2 \]

To find the \( x \)-intercept, we must solve 
\[ f(x) = 0. \]

Setting \( f(x) = 0 \), we must solve \( \sqrt{4 - 3x} = x \). Squaring, we must solve \( 4 - 3x = x^2 \). Adding \( 4 - 3x \) to both sides of the equation, we see that we must solve \( x^2 + 3x - 4 = 0 \). This polynomial factors as \( x^2 + 3x - 4 = (x - 1)(x + 4) \). As \( 0 \leq x \leq \frac{4}{3} \), \(-4\) is not an intercept. Moreover, substituting, we find that 1 is not an intercept either! (By squaring we introduced a potential sign error.)
In our case, the function is defined everywhere, but
\[ \lim_{x \to \frac{4}{3}} f'(x) = \lim_{x \to \frac{4}{3}} \left(1 - \frac{\frac{3}{2\sqrt{3-4x}}}{\frac{3}{2\sqrt{3-4x}}} \right) = \infty \] So, \( x = \frac{4}{3} \) is a vertical asymptote to the graph of \( f(x) \).