In sketching the graph of a function, one should look for basic qualitative features.

- Relative Extrema
- Inflection Points
- Intercepts
- Concavity
- Asymptotes
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Graphing a cubic

Example

Graph $f(x) = x^3 - 2x^2 + x - 1$ for $-3 \leq x \leq 3$. 
Finding the relative extrema: the endpoints

To find the relative extrema we look at the endpoints of the domain of the function (if any) and at the zeros of the derivative.

- \( f(-3) = -27 - 18 - 3 - 1 = -49 \)
- and
- \( f(3) = 27 - 18 + 3 - 1 = 11 \).
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We compute the first derivative
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\[ f'(x) = 3x^2 - 4x + 1 \]
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So that \( f'(x) = 0 \) for \( x = \frac{1}{3} \) and \( x = 1 \).

We evaluate the function at these points obtaining \( f\left(\frac{1}{3}\right) = \frac{1}{27} - \frac{6}{27} + \frac{9}{27} - \frac{27}{27} = \frac{-23}{27} \).
We compute the first derivative

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and

\[ f(1) = 1 - 2 + 1 - 1 = -1. \]
Concavity may change where 
$f''(x) = 0$. 
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\( f''(x) = 0 \).
In our case, we have \( f''(x) = 6x - 4 \) so that the only possible inflection point occurs at \( x = \frac{2}{3} \).
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In our case, we have \( f''(x) = 6x - 4 \) so that the only possible inflection point occurs at \( x = \frac{2}{3} \).
As \( 6x - 4 < 0 \) when \( x < \frac{2}{3} \) and \( 6x - 4 > 0 \) when \( x > \frac{2}{3} \), we see that the graph of \( f(x) \) is concave down for \( x < \frac{2}{3} \) and is concave up when \( x > \frac{2}{3} \) and that \( x = \frac{2}{3} \) is an inflection point.
To find the $y$-intercept we evaluate at zero:
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To find the $x$-intercept, we must solve $f(x) = 0$. 
Finding the intercepts

To find the $y$-intercept we evaluate at zero:

$$f(0) = -1$$

To find the $x$-intercept, we must solve $f(x) = 0$. In our case, there is no simple algebraic solution to this problem. However, we can approximate the location of the $x$-intercept by evaluating $f$ in the range $1 \leq x \leq 2$. 
Finding the asymptotes

To find the asymptotes, we look at undefined points for the function and at \( \lim_{x \to \pm \infty} f'(x) \).
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Finding the asymptotes

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In our case, the function is continuous at every real number. So, there are no vertical asymptotes.
As \( f(x) \) is defined only in the region \(-3 \leq x \leq 3\), we need not consider asymptotes at \( \infty \).
Example

Graph \( f(x) = x^2 - 2x + \frac{1}{x-1} \)
for \(-3 \leq x \leq 3\).
Finding the relative extrema: the endpoints

To find the relative extrema we look at the endpoints of the domain of the function (if any) and at the zeros of the derivative.

- \( f(-3) = (-3)^2 - 2(-3) + \frac{1}{(-3 - 1)} = 14.75 \) and
- \( f(3) = 3^2 - 2(3) + \frac{1}{(3 - 1)} = 3.5 \).
Finding the relative extrema: the endpoints

To find the relative extrema we look at the endpoints of the domain of the function (if any) and at the zeros of the derivative.

At the endpoints

1. \( f(-3) = (-3)^2 - 2(-3) + 1/(-3 - 1) = 14.75 \) and
2. \( f(3) = 3^2 - 2(3) + 1/(3 - 1) = 3.5. \)
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\[ f'(x) = 2x - 2 - (x - 1)^{-2} \]
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\[ = \frac{2(x - 1)^3 - 1}{(x - 1)^2} \]
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So that \( f'(x) = 0 \) for \( x = 1 + \sqrt[3]{\frac{1}{2}} \).
We compute the first derivative

\[ f'(x) = 2x - 2 - (x - 1)^{-2} \]

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So that \( f'(x) = 0 \) for \( x = 1 + \frac{\sqrt[3]{1}}{\sqrt[3]{2}} \).

We evaluate the function at this point obtaining
We compute the first derivative

\[ f'(x) = 2x - 2 - (x - 1)^{-2} \]

\[ = \frac{2(x - 1)^3 - 1}{(x - 1)^2} \]

So that \( f'(x) = 0 \) for \( x = 1 + \sqrt[3]{\frac{1}{2}}. \)

We evaluate the function at this point obtaining \( f(1 + \sqrt[3]{\frac{1}{2}}) \approx 0.889881575 \).
Concavity may change where \( f''(x) = 0 \).
Concavity may change where $f''(x) = 0$. In our case, we have $f''(x) = 2 + 2(x - 1)^{-3}$ so that the only possible inflection point occurs at $x = 0$. 
Concavity may change where $f''(x) = 0$. In our case, we have $f''(x) = 2 + 2(x - 1)^{-3}$ so that the only possible inflection point occur at $x = 0$. As $f''(x) < 0$ when $0 < x < 1$ and $f''(x) > 0$ when $x > 1$ or $x < 0$, we see that the graph of $f(x)$ is concave down for $0 < x < 1$ and is concave up when $x < 0$ or $x > 1$ and that $x = 0$ is an inflection point.
Finding the intercepts

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To find the $x$-intercept, we must solve $f(x) = 0$. In our case, there is no simple algebraic solution to this problem. However, we can approximate the location of the $x$-intercept by evaluating $f$ in the range $-0.4 \leq x \leq -0.3$. 
Finding the asymptotes

To find the asymptotes, we look at undefined points for the function and at \( \lim_{x \to \pm \infty} f'(x) \).
Finding the asymptotes

To find the asymptotes, we look at undefined points for the function and at \( \lim_{x \to \pm \infty} f'(x) \). In our case, the function is undefined at \( x = 1 \) and there is clearly a vertical asymptote there.
Finding the asymptotes

To find the asymptotes, we look at undefined points for the function and at \( \lim_{x \to \pm \infty} f'(x) \).

In our case, the function is undefined at \( x = 1 \) and there is clearly a vertical asymptote there.

As the function is only defined for \( -3 \leq x \leq 3 \), we need not consider asymptotes at infinity.
Graphing a function with fractional powers

Example

Graph $f(x) = x + \sqrt{4 - 3x}$ for $0 \leq x \leq \frac{4}{3}$. 

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Finding the relative extrema: the endpoints

To find the relative extrema we look at the endpoints of the domain of the function (if any) and at the zeros of the derivative.

- $f(0) = 2$ and
- $f\left(\frac{4}{3}\right) = 0$. 
Finding the relative extrema: the endpoints

To find the relative extrema we look at the endpoints of the domain of the function (if any) and at the zeros of the derivative.

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To find the relative extrema we look at the endpoints of the domain of the function (if any) and at the zeros of the derivative.

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To find the relative extrema we look at the endpoints of the domain of the function (if any) and at the zeros of the derivative.

At the endpoints
- \( f(0) = 2 \) and
- \( f\left(\frac{4}{3}\right) = 0 \).
We compute the first derivative
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\[ f'(x) = 1 - \frac{3}{2}(4 - 3x)^{-\frac{1}{2}} \]
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\[ f'(x) = 1 - \frac{3}{2} (4 - 3x)^{-\frac{1}{2}} \]

So that \( f'(x) = 0 \) for \( x = \frac{7}{12} \).
We compute the first derivative

\[ f'(x) = 1 - \frac{3}{2} (4 - 3x)^{-\frac{1}{2}} \]

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We compute the first derivative

$$f'(x) = 1 - \frac{3}{2} (4 - 3x)^{-\frac{1}{2}}$$

So that $f'(x) = 0$ for $x = \frac{7}{12}$. We evaluate the function at this point obtaining
We compute the first derivative

\[ f'(x) = 1 - \frac{3}{2} (4 - 3x)^{-\frac{1}{2}} \]

So that \( f'(x) = 0 \) for \( x = \frac{7}{12} \).

We evaluate the function at this point obtaining \( f \left( \frac{7}{12} \right) = \frac{25}{12} \).
Concavity may change where \( f''(x) = 0 \).
Concavity may change where \( f''(x) = 0 \).
In our case, we have \( f''(x) = -\frac{9}{4}(4 - 3x)^{-\frac{3}{2}} \).
Concavity may change where \( f''(x) = 0 \).
In our case, we have \( f''(x) = \frac{-9}{4}(4 - 3x)^{-3/2} \).
In the range where \( f''(x) \) is defined, \( f''(x) \) is always negative so that the graph of \( f(x) \) is always concave down. Note, however, that \( f''(x) \) tends to zero as \( x \) approaches \( \frac{4}{3} \).
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Finding the intercepts

To find the $y$-intercept we evaluate at zero: $f(0) = 2$

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Setting $f(x) = 0$, we must solve $\sqrt{4 - 3x} = x$. 
Finding the intercepts

To find the $y$-intercept we evaluate at zero: $f(0) = 2$

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Setting $f(x) = 0$, we must solve $\sqrt{4 - 3x} = x$. Squaring, we must solve $4 - 3x = x^2$. 

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Finding the intercepts

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Setting $f(x) = 0$, we must solve $\sqrt{4 - 3x} = x$. Squaring, we must solve $4 - 3x = x^2$. Adding $4 - 3x$ to both sides of the equation, we see that we must solve $x^2 + 3x - 4 = 0$. This polynomial factors as $x^2 + 3x - 4 = (x - 1)(x + 4)$. 
Finding the intercepts

To find the $y$-intercept we evaluate at zero: $f(0) = 2$

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Adding $4 - 3x$ to both sides of the equation, we see that we must solve $x^2 + 3x - 4 = 0$.

This polynomial factors as $x^2 + 3x - 4 = (x - 1)(x + 4)$. As $0 \leq x \leq \frac{4}{3}$, $-4$ is not an intercept.
Finding the intercepts

To find the $y$-intercept we evaluate at zero: $f(0) = 2$
To find the $x$-intercept, we must solve $f(x) = 0$.
Setting $f(x) = 0$, we must solve $\sqrt{4 - 3x} = x$. Squaring, we must solve $4 - 3x = x^2$.
Adding $4 - 3x$ to both sides of the equation, we see that we must solve $x^2 + 3x - 4 = 0$. This polynomial factors as $x^2 + 3x - 4 = (x - 1)(x + 4)$. As $0 \leq x \leq \frac{4}{3}$, $-4$ is not an intercept. Moreover, substituting, we find that $1$ is not an intercept either! (By squaring we introduced a potential sign error.)
Finding the asymptotes

In our case, the function is defined everywhere, but
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In our case, the function is defined everywhere, but
\[
\lim_{x \to 4} f'(x) = \lim_{x \to 4} \left(1 - \frac{3}{2\sqrt{3 - 4x}}\right) = \infty
\]
Finding the asymptotes

In our case, the function is defined everywhere, but \( \lim_{x \to \frac{4}{3}} f'(x) = \lim_{x \to \frac{4}{3}} (1 - \frac{3}{2\sqrt{3-x}}) = \infty \). So, \( x = \frac{4}{3} \) is a vertical asymptote to the graph of \( f(x) \).