We have already defined the tangent line to a curve at a point to be the limit of the secant lines. In this section we give a precise sense to this definition.
Tangent line from a slope and a point

A line is determined by its slope and one point on the line. Thus, in saying that the tangent line to the curve $C$ at the point $P$ is the limit of the secant lines $\overline{PQ}$ as the point $Q$ on $C$ approaches $P$, we mean that the slope of the tangent line is the limit of the slopes of the secant lines $\overline{PQ}$. 
Computing slopes of secant lines on graphs of functions

In the case that the curve $C$ is the graph of a function $f(x)$ and $P = (a, f(a))$, then the nearby points on $C$ have the form $(a + \Delta x, f(a + \Delta x))$ with $\Delta x \neq 0$ (but $\approx 0$).

In this case, the slope of the tangent line to $C$ at $P$ is by definition $f'(a)$ and we have formula for the slope of the secant line $\overline{PQ}$:

$$\frac{\Delta y}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

In terms of limits, our formula for the derivative of $f$ at $a$ is:

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$
Definition of the limit

Definition

If \( g(x) \) is a function and \( a < b < c \) are numbers for which the intervals \((a, b)\) and \((b, c)\) are in the domain of \( g \) and \( \ell \) is another real number, then we say that the limit of \( g(x) \) as \( x \) approaches \( b \) is \( \ell \), written \( \lim_{x \to b} g(x) = \ell \), if whenever \( x \approx b \) we have \( g(x) \approx \ell \).

More precisely, for any desired degree of accuracy, by taking \( |x - b| \) small enough, \( g(x) \) is equal to \( \ell \) to within the prescribed degree of accuracy.

If there is no real number \( \ell \) towards which \( g(x) \) approaches at \( x \) approaches \( b \), then we say that \( \lim_{x \to b} g(x) \) does not exist.

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Example

Find the following limits.

- $\lim_{x \to 20} 2x$
- $\lim_{x \to 0} \frac{1}{x^2}$

As $x$ approaches 20, $2x$ approaches 40. So, $\lim_{x \to 20} 2x = 40$.

The limit $\lim_{x \to 0} \frac{1}{x^2}$ does not exist:

If $\ell$ is any real number, then whenever $|x| < (|\ell| + 1)^{-\frac{1}{2}}$ we have $f(x) - \ell > 1$ so that $f(x)$ cannot possibly approach $\ell$. 
Derivatives as limits

To fit the equation

\[ f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} \]

into the general definition of a limit take \( g(x) = \frac{f(a + x) - f(a)}{x} \)
and \( b = 0 \).
Recalculating $\frac{d}{dx}(x^2)$

We return to our calculation of the derivative of $y = f(x) = x^2$. We computed that the slope of the secant line between $(a, f(a))$ and $(a + \Delta x, f(a + \Delta x))$ to be $\frac{\Delta y}{\Delta x} = 2a + \Delta x$. We concluded $f'(a) = 2a$ as $\lim_{\Delta x \to 0} 2a + \Delta x = 2a + 0 = 2a$. 
Formulae for limits

There are many useful rules for computing limits. In the above example we used a couple of these rules without explicitly mentioning them.

Let \( g(x) \) and \( h(x) \) be two functions defined on either side of the real number \( b \) for which the limits \( \lim_{x \to b} g(x) \) and \( \lim_{x \to b} h(x) \) exist.

- \( \lim_{x \to b} (g + h)(x) = \lim_{x \to b} g(x) + \lim_{x \to b} h(x) \)
- \( \lim_{x \to b} (g \cdot h)(x) = \left( \lim_{x \to b} g(x) \right) \left( \lim_{x \to b} h(x) \right) \)
- For any constant \( \alpha \), \( \lim_{x \to b} \alpha g(x) = \alpha \lim_{x \to b} g(x) \).
- For any real number \( r \), provided that \( \lim_{x \to b} g(x) > 0 \), we have \( \lim_{x \to b} (g(x))^r = (\lim_{x \to b} g(x))^r \)
- Provided that \( \lim_{x \to b} g(x) \neq 0 \), we have \( \lim_{x \to b} \frac{h(x)}{g(x)} = \frac{\lim_{x \to b} h(x)}{\lim_{x \to b} g(x)} \).
Using the limit rules

Example

Compute \( \lim_{x \to 25} 4x + \frac{1}{\sqrt[3]{x^2}} \).

\[
\lim_{x \to 25} (4x + \frac{1}{\sqrt[3]{x^2}}) = \lim_{x \to 25} (4x + x^{-\frac{3}{2}})
\]

\[
= 4 \lim_{x \to 25} x + \left( \lim_{x \to 25} x \right)^{-\frac{3}{2}}
\]

\[
= 4(25) + 25^{-\frac{3}{2}}
\]

\[
= 100 + \frac{1}{\sqrt[3]{25}^3}
\]

\[
= 100 + \frac{1}{5^3}
\]
Limits of polynomials

From the general rules for limits we have a formula for the limit of a polynomial.

For a polynomial \( p(x) = a_0 + a_1x + \ldots + a_nx^n \) and a real number \( b \),

\[
\lim_{x \to b} p(b) = \lim_{x \to b} (a_0 + a_1x + \ldots + a_nx^n) \\
= ( \lim_{x \to b} a_0 ) + ( \lim_{x \to b} a_1x ) + \ldots + ( \lim_{x \to b} a_nx^n ) \\
= a_0 + a_1(\lim_{x \to b} x) + \ldots + a_n(\lim_{x \to b} x)^n \\
= a_0 + a_1b + \ldots + a_nb^n \\
= p(b)
\]
An example of a limit of a polynomial

Example

Compute \( \lim_{x \to 8} 1 + x^3 + 5x^4 \).

To compute the limit we need only evaluate the polynomial. Thus,

\[
\lim_{x \to 8} 1 + x^3 + 5x^4 = 1 + 8^3 + 5(8^4)
\]

\[
= 1 + 512 + 5(4096)
\]

\[
= 20993
\]
Limits of rational functions

For rational functions, the calculation of a limit is more complicated.

Example

Compute \( \lim_{x \to 4} \frac{x^3 - 64}{x - 4} \).

The function \( \frac{x^3 - 64}{x - 4} \) is not defined at 4.

However, at every real number \( x \) other than four, we have

\[
\frac{x^3 - 64}{x - 4} = x^2 + 4x + 16
\]

\[
\lim_{x \to 4} \frac{x^3 - 64}{x - 4} = \lim_{x \to 4} (x^2 + 4x + 16)
\]
Another example of a limit of a rational function

Example

Compute \( \lim_{x \to 1} \frac{x^2 + 1}{x^2 - 1} \).

In this case, there is no limit. The numerator \( x^2 + 1 \) approaches 2 as \( x \) approaches 1 while the denominator approaches 0 as \( x \) approaches 1.
Limits at \( \infty \)

It also makes sense to compute limits as \( x \) approaches \( \pm \infty \).

**Definition**

For a function \( f(x) \) defined for \( x > N \) for some number \( N \) and a real number \( \ell \), then we say \( \lim_{x \to \infty} f(x) = \ell \) if for any degree of precision we have \( f(x) = \ell \) to within that degree of precision for all \( x \) sufficiently large.

Likewise,

**Definition**

For a function \( f(x) \) defined for \( x < N \) for some number \( N \) and a real number \( \ell \), then we say \( \lim_{x \to -\infty} f(x) = \ell \) if for any degree of precision we have \( f(x) = \ell \) to within that degree of precision for all \( x \) negative with \( |x| \) sufficiently large.
A limit at $\infty$

Example

Compute $\lim_{x \to \infty} \frac{6}{x + 1}$.

For $x$ very large, $x + 1$ is also very large so that $\frac{6}{x + 1}$ is very small (though positive). Thus, $\lim_{x \to \infty} \frac{6}{x + 1} = 0$. 
Recall that a function $f(x)$ is differentiable at the point $a$ just in case there is a unique, nonvertical tangent line to the graph of $f$ at $(a, f(a))$. 
Some examples of nondifferentiability

Example

- $f(x) = \sqrt[5]{x}$ at $a = 0$
- $f(x) = |x|$ at $a = 0$
- $f(x) = \begin{cases} 3x & \text{for } x < 0 \\ 5x & \text{for } x \geq 0 \end{cases}$ at $a = 0$
- $f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$ at $a = 0$
Nondifferentiability in practice

Nondifferentiable functions are not merely mathematical pathologies. Functions occurring in applications, especially when defined by cases, may be nondifferentiable.
Nondifferentiability in an application

Example

The sales staff at a certain retailer receive a base salary of thirty thousand dollars plus a commission of ten percent of their first fifty thousand dollars in sales and twenty-five percent of all their sales beyond fifty thousand dollars. Write a mathematical expression for the number of dollars $f(x)$ a salesperson will be paid for $x$ dollars in sales. Determine where $f(x)$ is differentiable.
Solution

For $x$ between 0 and 50,000, a salesperson earns $30,000 plus $.1x$. If $x \geq 50,000$, then the salesperson earns $30,000 + (.10)(50,000) = 35,000$ dollars for the first fifty thousand dollars of sales plus another $(0.25)(x - 50,000)$ dollars in commission for the sales above and beyond fifty thousand dollars. Thus,

$$f(x) = \begin{cases} 30,000 + (.10)x & \text{for } 0 \leq x \leq 50,000 \\ 35,000 + (.25)x & \text{for } x \geq 35,000 \end{cases}$$

The function $f$ is differentiable at every point in its domain except for $x = 35,000$. 

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Equivalent formulations of differentiability

- As an equivalent definition of **differentiable** we have \( f(x) \) is differentiable at \( a \) if and only if \( f'(x) \) is defined at \( a \).
- Analytically, \( f(x) \) is differentiable at \( a \) if and only if
  \[
  \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}
  \]
  exists.
- Graphically, differentiability corresponds to smoothness.
Sources of nondifferentiability

There are (at least) three major causes of nondifferentiability.

- a cusp \( f(x) = x^{\frac{2}{3}} \)
- a vertical tangent line \( f(x) = x^{\frac{1}{7}} \)
- a discontinuity \( f(x) = \begin{cases} x & \text{for } x < 0 \\ x + 1 & \text{for } x \geq 0 \end{cases} \)
Continuity

**Definition**

A function $f(x)$ is **continuous** at $a$ if $f(a) = \lim_{x \to a} f(x)$. The function $f(x)$ is **continuous** if it is continuous at every point in its domain.

Graphically, $f(x)$ is continuous if its graph can be drawn without lifting the pen.
Examples of (dis)continuity

Example

1. $f(x) = \begin{cases} 
  x^2 & \text{for } x \leq 2 \\
  2x + 2 & \text{for } x > 2 
\end{cases}$

2. $f(x) = \begin{cases} 
  x^3 - 1 & \text{for } x \leq 1 \\
  5x & \text{for } x > 1 
\end{cases}$
Differentiability implies continuity

Our result on the limits of polynomials implies that every polynomial is continuous. Moreover, every rational function is continuous wherever it is defined. As a general rule, if \( f(x) \) is differentiable at \( a \), then \( f(x) \) is continuous at \( a \).
Rules for differentiation

There are several useful rules for the calculation of derivatives. In this section, we will introduce three of them.
Scalar multiplication

**Theorem**

If \( f(x) \) is a function, \( \alpha \) is number, \( a \) is a number in the domain of \( f(x) \) and \( f(x) \) is differentiable, then \( \frac{d}{dx}(\alpha f(x)) = \alpha \frac{d}{dx} f(x) \).

**Proof.**

\[
\frac{d}{dx}(\alpha f)(a) = \lim_{\Delta x \to 0} \frac{\alpha f(a + \Delta x) - \alpha f(a)}{\Delta x} \\
= \lim_{\Delta x \to 0} \frac{\alpha (f(a + \Delta x) - f(a))}{\Delta x} \\
= \alpha \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} \\
= \alpha f'(a)
\]
Scalar multiplication in an example

Example

Compute $\frac{d}{dx}(5x^3)$

$$\frac{d}{dx}(5x^3) = 5\frac{d}{dx}(x^3)$$

$$= 5(3x^2)$$

$$= 15x^2$$
Sum rule

**Theorem**

If \( f(x) \) and \( g(x) \) are two differentiable functions, then so is 
\( (f + g)(x) \) and 
\( (f + g)'(x) = f'(x) + g'(x) \).

**Proof.**

\[
(f + g)'(a) = \lim_{\Delta x \to 0} \frac{(f + g)(a + \Delta x) - (f + g)(a)}{\Delta x} \\
= \lim_{\Delta x \to 0} \frac{f(a + \Delta x) + g(a + \Delta x) - f(a) - g(a)}{\Delta x} \\
= \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(a + \Delta x) - g(a)}{\Delta x} \\
= f'(a) + g'(a)
\]
Using the sum rule

Example

Compute the derivative of \( f(x) = 8x^{19} - 9\sqrt[4]{x} \).

\[
f'(x) = \frac{d}{dx} (8x^{19} - 9\sqrt[4]{x})
\]
\[
= \frac{d}{dx} (8x^{19} + (-9)\sqrt[4]{x})
\]
\[
= \frac{d}{dx} (8x^{19}) + \frac{d}{dx} (-9x^{1/4})
\]
\[
= 8\frac{d}{dx} (x^{19}) + (-9)\frac{d}{dx} (x^{1/4})
\]
\[
= (8)19x^{18} + (-9)\frac{1}{4}x^{-3}
\]
\[
= 152x^{18} - 9/(4\sqrt[4]{x^3})
\]
Theorem

If $f(x)$ is a differentiable function with $f(x) > 0$ and $r$ is a real number, then the function $g(x) = (f(x))^r$ is differentiable and $g'(x) = r(f(x))^{r-1}f'(x)$. 
Computing derivatives using the power rule

Example

Compute the derivative of $f(x) = \sqrt[4]{7x^2 + 1}$.

$$f'(x) = \frac{d}{dx}((7x^2 + 1)^{\frac{1}{4}})$$

$$= \frac{1}{4}(7x^2 + 1)^{-\frac{3}{4}} \frac{d}{dx}(7x^2 + 1)$$

$$= \frac{1}{4}(7x^2 + 1)^{-\frac{3}{4}} \left( \frac{d}{dx}(7x^2) + \frac{d}{dx}(1) \right)$$

$$= \frac{1}{4}(7x^2 + 1)^{-\frac{3}{4}} \left( 7 \frac{d}{dx}(x^2) + 0 \right)$$

$$= \frac{1}{4}(7x^2 + 1)^{-\frac{3}{4}} \left( (7)2x \right)$$

$$= \frac{7x}{(2^{\frac{4}{3}}/(7x^2 + 1)^{\frac{3}{4}})}$$
Another derivative computation

Example

Compute the derivative of \( f(x) = \frac{5}{\sqrt{8x^3 + 2x}} \).

\[
f'(x) = \frac{d}{dx} \left( \frac{5}{\sqrt{8x^3 + 2x}} \right)
= 5 \frac{d}{dx} (8x^3 + 2x)^{\frac{1}{2}}
= 5 \left( \frac{1}{2} \right) (8x^3 + 2x)^{-\frac{1}{2}} \frac{d}{dx} (8x^3 + 2x)
= \frac{5}{2} (8x^3 + 2x)^{-\frac{1}{2}} (24x^2 + 2)
= \frac{60x^2 + 5}{\sqrt{8x^3 + 2x}}
\]
While we conventionally write functions with an independent variable $x$ and set the result equal to $y$ (ie $y = f(x)$), it is not necessary to use this choice of variables.
Derivative computation

Example

Let \( f(t) = 3t^2 + 8 \). Compute \( f'(t) \).

If \( g(x) = 3x^2 + 8 \), then

\[
g'(x) = \frac{d}{dx}(3x^2 + 8) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(8) = 3\frac{d}{dx}(x^2) + 0 = (3)(2)x = 6x
\]
Alternate notation for derivatives

We have alternate notations for the derivative. If $y = f(x)$ is a differentiable function, then we may write the derivative of $f$ as

- $f'(x)$
- $y'$
- $\frac{d}{dx} f(x)$
- $\frac{dy}{dx}$
Alternate independent variables and derivatives

When the independent variable of our function is something other than $x$, then our notation for the derivative reflects this fact.

Example

If $x = s(t)$, then we may write the derivative of $s$ (with respect to $t$) as

- $s'(t)$
- $x'$
- $\frac{ds}{dt}$
- $\frac{dx}{dt}$
Example

Let \( h(y) = \frac{8}{y^3 + \sqrt{y}} \). Compute \( h'(y) \).

\[
\begin{align*}
h'(y) & = \frac{d}{dy} \left( \frac{8}{y^3 + \sqrt{y}} \right) \\
& = \frac{d}{dy} \left( 8 (y^3 + y^{1/2})^{-1} \right) \\
& = (8)(-1)(y^3 + y^{1/2})^{-2} \left( \frac{d}{dy} (y^3) + \frac{d}{dy} (y^{1/2}) \right) \\
& = -8 (y^3 + y^{1/2})^{-2} \left( 3y^2 + \frac{1}{2} y^{-1/2} \right) \\
& = -8 \left( 3y^2 + \frac{1}{2y^{1/2}} \right) \\
& = \frac{-8(3y^2 + \frac{1}{2\sqrt{y}})}{(y^3 + \sqrt{y})^2}.
\end{align*}
\]
Notation indicating the independent variable

For now, the notation $\frac{d}{dt}$, $\frac{d}{dx}$, $\frac{d}{dy}$, *et cetera* is used merely to indicate that the independent variable is $t$, $x$, $y$, respectively. When we deal with functions of several variables (in Math 16b), then this notation will carry the additional meaning that we regard all other variables as constant.
Derivative of quadratics

Example

Let \( a, b, \) and \( c \) be three real numbers. Let \( f(t) = at^2 + bt + c. \)
Compute \( f'(t). \)

\[
f'(t) = \frac{d}{dt}(at^2 + bt + c)
\]

\[
= \frac{d}{dt}(at^2) + \frac{d}{dt}(bt) + \frac{d}{dt}(c)
\]

\[
= a \frac{d}{dt}(t^2) + b \frac{d}{dt}(t) + 0
\]

\[
= a(2t) + b(1) + 0
\]

\[
= 2at + b
\]
The second derivative

**Definition**

If $f(x)$ is a differentiable function of $x$, then $f'(x)$ is another function of $x$. The derivative of $f'(x)$, denoted $f''(x)$, is called the **second derivative** of $f$. We call $f'(x)$ the **first derivative** of $f$. 

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As with the derivative itself, there are alternate notations for the second derivative. If $y = f(x)$, then we may denote the second derivative of $f$ by:

- $f''(x)$
- $y''$
- $f^{(2)}(x)$
- $\frac{d^2}{dx^2} f(x)$
- $\frac{d^2 y}{dx^2}$
Computing a second derivative

Example

Let \( f(t) = 4t + \frac{8}{\sqrt[3]{t}} \). Compute \( f''(t) \).

We start by computing \( f'(t) \).

\[
f'(t) = \frac{d}{dt} \left( 4t + \frac{8}{\sqrt[3]{t}} \right) = \frac{d}{dt} (4t) + \frac{d}{dt} \left( 8t^{-\frac{1}{3}} \right) = 4 + 8 \left( -\frac{1}{3} \right) t^{-\frac{4}{3}} = 4 - \frac{8}{3^{\frac{3}{2}}t^{\frac{4}{2}}}.
\]
Computation continued

\[ f''(t) = \frac{d}{dt} f'(t) \]

\[ = \frac{d}{dt} \left( 4 - \frac{8}{3 \sqrt[3]{t^4}} \right) \]

\[ = \frac{d}{dt} (4) - \frac{8}{3} \frac{d}{dt} \left( t^{-\frac{4}{3}} \right) \]

\[ = 0 - \frac{8}{3} \cdot \frac{-4}{3} t^{-\frac{7}{3}} \]

\[ = \frac{32}{9 \sqrt[3]{t^7}} \]
Another computation of a second derivative

Example

Let $x = f(y) = \frac{1}{y}$. Compute $\frac{d^2x}{dy^2}$.

\[
\frac{d^2x}{dy^2} = \frac{d}{dy} \left( \frac{dx}{dy} \right) \\
= \frac{d}{dy} \left( \frac{1}{y} \right) \\
= \frac{d}{dy} (y^{-1}) \\
= -y^{-2} \\
= 2y^{-3}
\]
If $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ is a function of $x$. If $a$ is a real number in the domain of $f'(x)$, then we write $\left.\frac{dy}{dx}\right|_{x=a}$ for $f'(a)$, the result of evaluating the derivative of $f$ at $a$. Likewise, we write $\left.\frac{d^2y}{dx^2}\right|_{x=a}$ for $f''(a)$. 

Notation for evaluation of derivatives
Evaluating a second derivative

Example

Let \( y = g(x) = x^2 - \sqrt{x} \). Compute \( \frac{d^2}{dx^2} g \big|_{x=4} \).

By definition, \( \frac{d^2}{dx^2} g \big|_{x=4} = g''(4) \). Thus, we need only compute \( g''(x) \) and then evaluate at 4.
Computation continued

\[ g''(x) = \frac{d^2}{dx^2}(x^2 - \sqrt{x}) \]

\[ = \frac{d}{dx}\left( \frac{d}{dx}(x^2 - x^{1/2}) \right) \]

\[ = \frac{d}{dx}\left(2x - \frac{1}{2}x^{-1/2}\right) \]

\[ = 2 - \frac{1}{2}(-1)\left(-\frac{3}{2}\right)x^{-3/2} \]

\[ = 2 + \frac{1}{4\sqrt{x^3}} \]
Evaluating

Evaluating at $x = 4$, we conclude

$$\frac{d^2}{dx^2} g \bigg|_{x=4} = g''(4)$$

$$= 2 + \frac{1}{4 \sqrt{4^3}}$$

$$= 2 + \frac{1}{4(8)}$$

$$= 2 + \frac{1}{32}$$

$$= \frac{65}{32}$$
We can interpret the derivative of a function as a rate of change. If $x_{\text{initial}}$ is some initial value of $x$ and $x_{\text{final}}$ is the final value of $x$, then we can regard $\Delta x := x_{\text{final}} - x_{\text{initial}}$ as a change in $x$. If $y = f(x)$ is a function with $x_{\text{initial}}$ and $x_{\text{final}}$ in the domain of $f$, then

\[
\Delta y := f(x_{\text{final}}) - f(x_{\text{initial}}) = f(x_{\text{initial}} + \Delta x) - f(x_{\text{initial}})
\]

is the change in the value of the function from the initial point to the final point.

The quotient $\frac{\Delta y}{\Delta x}$ may be regarded as the average rate of change of $f(x)$ from $x_{\text{initial}}$ to $x_{\text{final}} = x_{\text{initial}} + \Delta x$. 
A computation of an average rate of change

Example

Let \( f(x) = x^3 + 1 \). Compute the average rate of change from \( x = 0.5 \) to \( x = 0.6 \).

We compute \( f(0.6) = (0.6)^3 + 1 = 1.216 \) and \( f(0.5) = (0.5)^3 + 1 = 1.125 \). So \( \Delta y = 1.216 - 1.125 = 0.091 \). Of course, \( \Delta x = 0.6 - 0.5 = 0.1 \).

So, the average rate of change of \( f(x) = x^3 + 1 \) from 0.5 to 0.6 is

\[
\frac{\Delta y}{\Delta x} = \frac{0.091}{0.1} = 0.91
\]
Rate of change as slope of secant line

Our expression for the average rate of change of a function is identical to our formula for the slope of the secant line between \((x_{\text{initial}}, f(x_{\text{initial}}))\) and \((x_{\text{final}}, f(x_{\text{final}}))\).
Another expression for the average rate of change

If we set $a := x_{\text{initial}}$ and $h := \Delta x$, so that $x_{\text{final}} = a + h$, then the average rate of change of the function $y = f(x)$ from $a$ to $a + h$ is

$$\frac{f(a + h) - f(a)}{h}$$
Derivative as instantaneous rate of change

If \( f(x) \) is differentiable at \( a \), then

\[
f'(a) = \lim_{{h \to 0}} \frac{{f(a + h) - f(a)}}{h}
\]

may be regarded as the **instantaneous rate of change** of \( f \) at \( a \).
Computing an instantaneous rate of change

Example

Let $f(x) = x^3 + 1$. Compute the instantaneous rate of change of $f$ at $a = 0.5$.

We compute $f'(x) = 3x^2$. Thus, the instantaneous rate of change of $f(x)$ at $a = 0.5$ is $3(0.5)^2 = 0.75$. 
The velocity of an object may be computed as a derivative. The average speed of an object is the total distance traveled divided by the time elapsed.

If we denote by \( x(t) \) the position of an object (measured as the distance to the right of some fixed point) at time \( t \), then rate of change of \( x \) is the velocity of the object.

**Definition**

The velocity of an object is the instantaneous rate of change of its positive. That is, if its position is given by \( x(t) \), then its velocity is \( v(t) = x'(t) \).
Warning on speed and velocity

Warning

The **speed** of an object is the magnitude of its **velocity**. An object which is moving backwards has a negative velocity, but its speed can never be negative.
Computing velocity

Example

An object dropped from a height one thousand feet in a vacuum has a position of \( x(t) = 1000 - 16t^2 \) after \( t \) seconds. What is its velocity after ten seconds?

\[ v(t) = x'(t) = -32t. \]

Thus, the velocity of the object after ten seconds is \(-320\) feet per second.
Acceleration as a second derivative

**Definition**

The rate of change of the velocity of an object is its **acceleration**. That is, if $v(t)$ denotes the velocity of the object at time $t$, then its acceleration is $a(t) = v'(t)$.

The velocity of an object having position $x(t)$ is $v(t) = x'(t)$. Thus, the acceleration of that object if $a(t) = v'(t) = x''(t)$.
Computing acceleration

Example

In the above example of the object dropped in a vacuum, compute its acceleration after ten seconds.

We already know that the velocity is $v(t) = -32t$. Thus, the acceleration is $a(t) = v'(t) = -32$. Therefore, after ten seconds the acceleration of $-32$ feet per second per second.
\[ \Delta y \text{ in terms of } \Delta x \]

By definition,

\[ [ \text{rate of change}] = \frac{\Delta y}{\Delta x} \]

Multiplying by \( \Delta x \), we obtain

\[ [\text{rate of change}] \Delta x = \Delta y \]

The derivative is well-approximated by the slopes of the secant lines. Thus, we have

\[ \frac{dy}{dx} \Delta x \approx \Delta y \]
Approximating a change using derivative

Example

Let \( p(t) \) denote the number of pages read after \( t \) minutes. If \( p(60) = 50 \) and \( p'(60) = 0.5 \), estimate the number of pages read after one hour and five minutes.

\[ p'(t) \] is the rate at which the pages are read. We estimate

\[ p(65) \approx p(60) + 5p'(60) = 50 + 5(0.5) = 52.5. \]
Marginal revenue

In economics, derivatives are often introduced through the term *marginal*. For example, if \( R(x) \) is the revenue generated by the application of \( x \) units of capital, then the *marginal revenue* is the additional revenue generated by an additional unit of capital.
Example

Suppose that the profit from a certain enterprise as a function of the number \( x \) of worker-hours of labor is given by

\[
P(x) = 100(x + \frac{1}{x}).
\]

What is the marginal profit at 8 worker-hours?

\[
P'(x) = 100(1 - x^{-2})\]

Thus,

\[
P'(8) = 100\left(1 - \frac{1}{64}\right) = \frac{6300}{64}.
\]