Math 16A (Autumn 2005)

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Week 3
We have already defined the tangent line to a curve at a point to be the limit of the secant lines. In this section we give a precise sense to this definition.
A line is determined by its slope and one point on the line.
A line is determined by its slope and one point on the line. Thus, in saying that the tangent line to the curve $C$ at the point $P$ is the limit of the secant lines $\overline{PQ}$ as the point $Q$ on $C$ approaches $P$, we mean that the slope of the tangent line is the limit of the slopes of the secant lines $\overline{PQ}$. 
In the case that the curve $C$ is the graph of a function $f(x)$ and $P = (a, f(a))$, then the nearby points on $C$ have the form $(a + \Delta x, f(a + \Delta x))$ with $\Delta x \neq 0$ (but $\approx 0$).
In the case that the curve $C$ is the graph of a function $f(x)$ and
$P = (a, f(a))$, then the nearby points on $C$ have the form
$(a + \Delta x, f(a + \Delta x))$ with $\Delta x \neq 0$ (but $\approx 0$).
In this case, the slope of the tangent line to $C$ at $P$ is by definition
$f'(a)$ and we have formula for the slope of the secant line $PQ$:

$$\frac{\Delta y}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{\Delta x}$$
In the case that the curve \( C \) is the graph of a function \( f(x) \) and \( P = (a, f(a)) \), then the nearby points on \( C \) have the form \((a + \Delta x, f(a + \Delta x))\) with \( \Delta x \neq 0 \) (but \( \approx 0 \)). In this case, the slope of the tangent line to \( C \) at \( P \) is by definition \( f'(a) \) and we have formula for the slope of the secant line \( \overline{PQ} \):

\[
\frac{\Delta y}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{\Delta x}
\]

In terms of limits, our formula for the derivative of \( f \) at \( a \) is:

\[
f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}
\]
Definition of the limit

Definition

If $g(x)$ is a function and $a < b < c$ are numbers for which the intervals $(a, b)$ and $(b, c)$ are in the domain of $g$ and $\ell$ is another real number, then we say that the limit of $g(x)$ as $x$ approaches $b$ is $\ell$, written $\lim_{x \to b} g(x) = \ell$, if whenever $x \approx b$ we have $g(x) \approx \ell$. 

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If \( g(x) \) is a function and \( a < b < c \) are numbers for which the intervals \((a, b)\) and \((b, c)\) are in the domain of \( g \) and \( l \) is another real number, then we say that the limit of \( g(x) \) as \( x \) approaches \( b \) is \( l \), written \( \lim_{x \to b} g(x) = l \), if whenever \( x \approx b \) we have \( g(x) \approx l \).

More precisely, for any desired degree of accuracy, by taking \( |x - b| \) small enough, \( g(x) \) is equal to \( l \) to within the prescribed degree of accuracy.
Definition of the limit

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If \( g(x) \) is a function and \( a < b < c \) are numbers for which the intervals \( (a, b) \) and \( (b, c) \) are in the domain of \( g \) and \( \ell \) is another real number, then we say that the limit of \( g(x) \) as \( x \) approaches \( b \) is \( \ell \), written \( \lim_{{x \to b}} g(x) = \ell \), if whenever \( x \approx b \) we have \( g(x) \approx \ell \).

More precisely, for any desired degree of accuracy, by taking \( |x - b| \) small enough, \( g(x) \) is equal to \( \ell \) to within the prescribed degree of accuracy.

If there is no real number \( \ell \) towards which \( g(x) \) approaches at \( x \) approaches \( b \), then we say that \( \lim_{{x \to b}} g(x) \) does not exist.
Examples of limits

Example

Find the following limits.

- \( \lim_{x \to 20} 2x \)
- \( \lim_{x \to 0} \frac{1}{x^2} \)
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As \( x \) approaches 20, \( 2x \) approaches 40. So, \( \lim_{x \to 20} 2x = 40 \).
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Examples of limits

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As \( x \) approaches 20, \( 2x \) approaches 40. So, \( \lim_{x \to 20} 2x = 40 \).

The limit \( \lim_{x \to 0} \frac{1}{x^2} \) does not exist:
Examples of limits

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Find the following limits.

- \( \lim_{x \to 20} 2x \)
- \( \lim_{x \to 0} \frac{1}{x^2} \)

As \( x \) approaches 20, \( 2x \) approaches 40. So, \( \lim_{x \to 20} 2x = 40 \).

The limit \( \lim_{x \to 0} \frac{1}{x^2} \) does not exist:

If \( \ell \) is any real number, then whenever \( |x| < (|\ell| + 1) \frac{1}{2} \) we have \( f(x) - \ell > 1 \) so that \( f(x) \) cannot possibly approach \( \ell \).
Derivatives as limits

To fit the equation

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

into the general definition of a limit take $g(x) = \frac{f(a + x) - f(a)}{x}$ and $b = 0$. 
We return to our calculation of the derivative of $y = f(x) = x^2$. 
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We return to our calculation of the derivative of $y = f(x) = x^2$. We computed that the slope of the secant line between $(a, f(a))$ and $(a + \Delta x, f(a + \Delta x))$ to be $\frac{\Delta y}{\Delta x} = 2a + \Delta x$. 
Recalculating $\frac{d}{dx}(x^2)$

We return to our calculation of the derivative of $y = f(x) = x^2$. We computed that the slope of the secant line between $(a, f(a))$ and $(a + \Delta x, f(a + \Delta x))$ to be $\frac{\Delta y}{\Delta x} = 2a + \Delta x$. We concluded $f'(a) = 2a$ as $\lim_{\Delta x \to 0} 2a + \Delta x = 2a + 0 = 2a$.  

Formulae for limits

There are many useful rules for computing limits. In the above example we used a couple of these rules without explicitly mentioning them.

- \( \lim_{x \to b} (g + h)(x) = \lim_{x \to b} g(x) + \lim_{x \to b} h(x) \)
- \( \lim_{x \to b} (g \cdot h)(x) = (\lim_{x \to b} g(x))(\lim_{x \to b} h(x)) \)
- For any constant \( \alpha \), \( \lim_{x \to b} \alpha g(x) = \alpha \lim_{x \to b} g(x) \).
- For any real number \( r \), provided that \( \lim_{x \to b} g(x) > 0 \), we have \( \lim_{x \to b} (g(x))^r = (\lim_{x \to b} g(x))^r \).
- Provided that \( \lim_{x \to b} g(x) \neq 0 \), we have \( \lim_{x \to b} (h/g)(x) = \frac{\lim_{x \to b} h(x)}{\lim_{x \to b} g(x)} \).
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Let $g(x)$ and $h(x)$ be two functions defined on either side of the real number $b$ for which the limits $\lim_{x \to b} g(x)$ and $\lim_{x \to b} h(x)$ exist.

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- For any constant $\alpha$, $\lim_{x \to b} \alpha g(x) = \alpha \lim_{x \to b} g(x)$.
- For any real number $r$, provided that $\lim_{x \to b} g(x) > 0$, we have $\lim_{x \to b} (g(x))^r = (\lim_{x \to b} g(x))^r$
- Provided that $\lim_{x \to b} g(x) \neq 0$, we have $\lim_{x \to b} \frac{h(x)}{g(x)} = \frac{\lim_{x \to b} h(x)}{\lim_{x \to b} g(x)}$. 
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Using the limit rules

Example

Compute \( \lim_{x \to 25} 4x + \frac{1}{\sqrt[3]{x^2}}. \)
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\lim_{x \to 25} (4x + \frac{1}{\sqrt[3]{x^3}}) = \lim_{x \to 25} (4x + x^{-\frac{3}{2}})
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Compute \( \lim_{x \to 25} 4x + \frac{1}{\sqrt[3]{x^2}}. \)

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\lim_{x \to 25} \left( 4x + \frac{1}{\sqrt[3]{x^2}} \right) = \lim_{x \to 25} \left( 4x + x^{-\frac{3}{2}} \right)
\]

\[
= 4 \lim_{x \to 25} x + \left( \lim_{x \to 25} x \right)^{-\frac{3}{2}}
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\[
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\]

\[
= 4(25) + 25^{-\frac{3}{2}}
\]
Using the limit rules

Example

Compute \( \lim_{{x \to 25}} (4x + \frac{1}{\sqrt[3]{x^2}}) \).

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\lim_{{x \to 25}} (4x + \frac{1}{\sqrt[3]{x^2}}) = \lim_{{x \to 25}} (4x + x^{\frac{-3}{2}}) \\
= 4 \lim_{{x \to 25}} x + (\lim_{{x \to 25}} x)^{\frac{-3}{2}} \\
= 4(25) + 25^{\frac{-3}{2}} \\
= 100 + \frac{1}{(\sqrt{25})^3}
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= 100 + \frac{1}{(\sqrt{25})^3}
\]
\[
= 100 + \frac{1}{5^3}
\]
Using the limit rules

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From the general rules for limits we have a formula for the limit of a polynomial.
Limits of polynomials

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$$\lim_{x \to b} p(b) = \lim_{x \to b} (a_0 + a_1x + \ldots + a_nx^n)$$
$$= (\lim_{x \to b} a_0) + (\lim_{x \to b} a_1x) + \ldots + (\lim_{x \to b} a_nx^n)$$
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= a_0 + a_1(\lim_{x \to b} x) + \ldots + a_n(\lim_{x \to b} x)^n
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= a_0 + a_1b + \ldots + a_nb^n
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$$= a_0 + a_1(\lim_{x \to b} x) + \ldots + a_n(\lim_{x \to b} x)^n$$

$$= a_0 + a_1b + \ldots + a_nb^n$$

$$= p(b)$$
Example

Compute \( \lim_{x \to 8} 1 + x^3 + 5x^4 \).
An example of a limit of a polynomial

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Compute \( \lim_{x \to 8} 1 + x^3 + 5x^4 \).

To compute the limit we need only evaluate the polynomial.
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To compute the limit we need only evaluate the polynomial. Thus,

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\lim_{x \to 8} 1 + x^3 + 5x^4 = 1 + 8^3 + 58^4
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Example

Compute \( \lim_{x \to 8} 1 + x^3 + 5x^4 \).

To compute the limit we need only evaluate the polynomial. Thus,

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\lim_{x \to 8} 1 + x^3 + 5x^4 = 1 + 8^3 + 5(4096) = 1 + 512 + 5(4096)
\]
Example

Compute \( \lim_{{x \to 8}} 1 + x^3 + 5x^4 \).

To compute the limit we need only evaluate the polynomial. Thus,

\[
\begin{align*}
\lim_{{x \to 8}} 1 + x^3 + 5x^4 &= 1 + 8^3 + 58^4 \\
&= 1 + 512 + 5(4096) \\
&= 20993
\end{align*}
\]
For rational functions, the calculation of a limit is more complicated.
Limits of rational functions

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Example

Compute \( \lim_{x \to 4} \frac{x^3 - 64}{x - 4} \).
Limits of rational functions

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Example

Compute \( \lim_{{x \to 4}} \frac{x^3 - 64}{x - 4} \).

The function \( \frac{x^3 - 64}{x - 4} \) is not defined at 4.
Limits of rational functions

For rational functions, the calculation of a limit is more complicated.

Example

Compute \( \lim_{x \to 4} \frac{x^3 - 64}{x - 4} \).

The function \( \frac{x^3 - 64}{x - 4} \) is not defined at 4. However, at every real number \( x \) other than four, we have

\[
\frac{x^3 - 64}{x - 4} = x^2 + 4x + 16
\]
Limits of rational functions

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\[ \lim_{x \to 4} \frac{x^3 - 64}{x - 4} = \lim_{x \to 4} (x^2 + 4x + 16) \]
For rational functions, the calculation of a limit is more complicated.

**Example**

Compute $\lim_{x \to 4} \frac{x^3 - 64}{x - 4}$.

The function $\frac{x^3 - 64}{x - 4}$ is not defined at 4. However, at every real number $x$ other than four, we have

$$\frac{x^3 - 64}{x - 4} = x^2 + 4x + 16$$

$$\lim_{x \to 4} \frac{x^3 - 64}{x - 4} = \lim_{x \to 4} x^2 + 4x + 16$$
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\]
Another example of a limit of a rational function

Example

Compute \( \lim_{x \to 1} \frac{x^2 + 1}{x^2 - 1} \).
Another example of a limit of a rational function

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Compute \( \lim_{x \to 1} \frac{x^2 + 1}{x^2 - 1} \).

In this case, there is no limit.
Another example of a limit of a rational function

**Example**

Compute \( \lim_{{x \to 1}} \frac{x^2 + 1}{x^2 - 1} \).

In this case, there is no limit. The numerator \( x^2 + 1 \) approaches 2 as \( x \) approaches 1 while the denominator approaches 0 as \( x \) approaches 1.
It also makes sense to compute limits as $x$ approaches $\pm\infty$. 
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**Definition**

For a function $f(x)$ defined for $x > N$ for some number $N$ and a real number $\ell$, then we say $\lim_{x \to \infty} f(x) = \ell$ if for any degree of precision we have $f(x) = \ell$ to within that degree of precision for all $x$ sufficiently large.
Limits at $\infty$

It also makes sense to compute limits as $x$ approaches $\pm\infty$.

**Definition**

For a function $f(x)$ defined for $x > N$ for some number $N$ and a real number $\ell$, then we say $\lim_{x \to \infty} f(x) = \ell$ if for any degree of precision we have $f(x) = \ell$ to within that degree of precision for all $x$ sufficiently large.

Likewise,

**Definition**

For a function $f(x)$ defined for $x < N$ for some number $N$ and a real number $\ell$, then we say $\lim_{x \to -\infty} f(x) = \ell$ if for any degree of precision we have $f(x) = \ell$ to within that degree of precision for all $x$ negative with $|x|$ sufficiently large.
Example

Compute \( \lim_{{x \to \infty}} \frac{6}{x + 1} \).
A limit at \( \infty \)

**Example**

Compute \( \lim_{x \to \infty} \frac{6}{x + 1} \).

For \( x \) very large, \( x + 1 \) is also very large so that \( \frac{6}{x + 1} \) is very small (though positive).
A limit at $\infty$

<table>
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<tr>
<td>Compute $\lim_{x \to \infty} \frac{6}{x + 1}$.</td>
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For $x$ very large, $x + 1$ is also very large so that $\frac{6}{x + 1}$ is very small (though positive). Thus, $\lim_{x \to \infty} \frac{6}{x + 1} = 0$.
Recall that a function $f(x)$ is differentiable at the point $a$ just in case there is a unique, nonvertical tangent line to the graph of $f$ at $(a, f(a))$. 
Some examples of nondifferentiability

Example

- \( f(x) = \sqrt[5]{x} \) at \( a = 0 \)
- \( f(x) = |x| \) at \( a = 0 \)
- \( f(x) = \begin{cases} 
3x & \text{for } x < 0 \\
5x & \text{for } x \geq 0 
\end{cases} \) at \( a = 0 \)
- \( f(x) = \begin{cases} 
0 & \text{for } x < 0 \\
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Some examples of nondifferentiability

Example

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Nondifferentiability in practice

Nondifferentiable functions are not merely mathematical pathologies. Functions occurring in applications, especially when defined by cases, may be nondifferentiable.
Example

The sales staff at a certain retailer receive a base salary of thirty thousand dollars plus a commission of ten percent of their first fifty thousand dollars in sales and twenty-five percent of all their sales beyond fifty thousand dollars.

Write a mathematical expression for the number of dollars \( f(x) \) a salesperson will be paid for \( x \) dollars in sales. Determine where \( f(x) \) is differentiable.
For $x$ between 0 and 50,000, a salesperson earns $30,000 plus $.1x.$
Solution

For $x$ between 0 and 50,000, a salesperson earns $30,000 plus $0.1x$. If $x \geq 50,000$, then the salesperson earns

$30,000 + (0.10)(50,000) = 35,000$ dollars for the first fifty thousand dollars of sales plus another $(0.25)(x - 50,000)$ dollars in commission for the sales above and beyond fifty thousand dollars.
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\[
30,000 + (.10)(50,000) = 35,000
\]
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$30,000 + (.10)(50,000) = 35,000$ dollars for the first fifty thousand dollars of sales plus another $(0.25)(x - 50,000)$ dollars in commission for the sales above and beyond fifty thousand dollars.

Thus,

$$f(x) = \begin{cases} 
30,000 + (.10)x & \text{for } 0 \leq x \leq 50,000 \\
35,000 + (.25)x & \text{for } x \geq 35,000 
\end{cases}$$
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$$f(x) = \begin{cases} 
30,000 + (0.10)x & \text{for } 0 \leq x \leq 50,000 \\
35,000 + (0.25)x & \text{for } x \geq 35,000
\end{cases}$$

The function $f$ is differentiable at every point in its domain except for $x = 35,000$. 
Equivalent formulations of differentiability

- As an equivalent definition of differentiable we have \( f(x) \) is differentiable at \( a \) if and only if \( f'(x) \) is defined at \( a \).
- Analytically, \( f(x) \) is differentiable at \( a \) if and only if
  \[
  \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}
  \]
  exists
- Graphically, differentiability corresponds to smoothness.
Equivalent formulations of differentiability

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Graphically, differentiability corresponds to smoothness.
There are (at least) three major causes of nondifferentiability.

- A cusp: \( f(x) = x^{\frac{2}{3}} \)
- A vertical tangent line: \( f(x) = x^{\frac{1}{7}} \)
- A discontinuity: \( f(x) = \begin{cases} x & \text{for } x < 0 \\ x + 1 & \text{for } x \geq 0 \end{cases} \)
Sources of nondifferentiability

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Definition

A function $f(x)$ is **continuous** at $a$ if $f(a) = \lim_{{x \to a}} f(x)$. The function $f(x)$ is **continuous** if it is continuous at every point in its domain.
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A function \( f(x) \) is \textit{continuous} at \( a \) if \( f(a) = \lim_{x \to a} f(x) \). The function \( f(x) \) is \textit{continuous} if it is continuous at every point in its domain.

Graphically, \( f(x) \) is continuous if its graph can be drawn without lifting the pen.
Examples of (dis)continuity

Example

\[ f(x) = \begin{cases} 
  x^2 & \text{for } x \leq 2 \\
  2x + 2 & \text{for } x > 2 
\end{cases} \]

\[ f(x) = \begin{cases} 
  x^3 - 1 & \text{for } x \leq 1 \\
  5x & \text{for } x > 1 
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Our result on the limits of polynomials implies that every polynomial is continuous. Moreover, every rational function is continuous wherever it is defined.
Our result on the limits of polynomials implies that every polynomial is continuous. Moreover, every rational function is continuous wherever it is defined. As a general rule, if \( f(x) \) is differentiable at \( a \), then \( f(x) \) is continuous at \( a \).
There are several useful rules for the calculation of derivatives. In this section, we will introduce three of them.
**Theorem**

If $f(x)$ is a function, $\alpha$ is number, $a$ is a number in the domain of $f(x)$ and $f(x)$ is differentiable, then

$$\frac{d}{dx} \left( \alpha f(x) \right) = \alpha \frac{d}{dx} f(x).$$
Scalar multiplication

Theorem

If \( f(x) \) is a function, \( \alpha \) is number, \( a \) is a number in the domain of \( f(x) \) and \( f(x) \) is differentiable, then \( \frac{d}{dx}(\alpha f(x)) = \alpha \frac{d}{dx} f(x) \).

Proof.

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Math 16A (Autumn 2005)
Scalar multiplication

Theorem

If $f(x)$ is a function, $\alpha$ is number, $a$ is a number in the domain of $f(x)$ and $f(x)$ is differentiable, then $\frac{d}{dx}(\alpha f(x)) = \alpha \frac{d}{dx}f(x)$.

Proof.

$$\frac{d}{dx}(\alpha f(a)) = \lim_{\Delta x \to 0} \frac{\alpha f(a + \Delta x) - \alpha f(a)}{\Delta x}$$
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**Proof.**

\[
\frac{d}{dx}(\alpha f)(a) = \lim_{\Delta x \to 0} \frac{\alpha f(a + \Delta x) - \alpha f(a)}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{\alpha(f(a + \Delta x) - f(a))}{\Delta x}
\]
Scalar multiplication

**Theorem**

If \( f(x) \) is a function, \( \alpha \) is number, \( a \) is a number in the domain of \( f(x) \) and \( f(x) \) is differentiable, then
\[
\frac{d}{dx}(\alpha f(x)) = \alpha \frac{d}{dx}f(x).
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**Proof.**

\[
\frac{d}{dx}(\alpha f(a)) = \lim_{\Delta x \to 0} \frac{\alpha f(a + \Delta x) - \alpha f(a)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\alpha(f(a + \Delta x) - f(a))}{\Delta x} = \alpha \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}
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If $f(x)$ is a function, $\alpha$ is number, $a$ is a number in the domain of $f(x)$ and $f(x)$ is differentiable, then $\frac{d}{dx}(\alpha f(x)) = \alpha \frac{d}{dx} f(x)$.

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\[
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\]
Scalar multiplication in an example

Example

Compute \( \frac{d}{dx}(5x^3) \)
Scalar multiplication in an example

Example

Compute \( \frac{d}{dx} (5x^3) \)

\[ \frac{d}{dx} (5x^3) = 5 \frac{d}{dx} (x^3) \]
Scalar multiplication in an example

Example

Compute \( \frac{d}{dx}(5x^3) \)

\[
\frac{d}{dx}(5x^3) = 5 \frac{d}{dx}(x^3) = 5(3x^2)
\]
Scalar multiplication in an example

Example

Compute \( \frac{d}{dx}(5x^3) \)

\[
\frac{d}{dx}(5x^3) = 5 \frac{d}{dx}(x^3) \\
= 5(3x^2) \\
= 15x^2
\]
Theorem

If \( f(x) \) and \( g(x) \) are two differentiable functions, then so is 
\((f + g)(x)\) and 
\[(f + g)'(x) = f'(x) + g'(x).\]
Theorem

If $f(x)$ and $g(x)$ are two differentiable functions, then so is $(f + g)(x)$ and $(f + g)'(x) = f'(x) + g'(x)$.

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Theorem

If \( f(x) \) and \( g(x) \) are two differentiable functions, then so is \((f + g)(x)\) and \((f + g)'(x) = f'(x) + g'(x)\).

Proof.

\[
(f + g)'(a) = \lim_{\Delta x \to 0} \frac{(f + g)(a + \Delta x) - (f + g)(a)}{\Delta x}
\]
Theorem

If \( f(x) \) and \( g(x) \) are two differentiable functions, then so is \((f + g)(x)\) and \((f + g)'(x) = f'(x) + g'(x)\).

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\[
= \lim_{\Delta x \to 0} \frac{f(a + \Delta x) + g(a + \Delta x) - f(a) - g(a)}{\Delta x}
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Theorem

If \( f(x) \) and \( g(x) \) are two differentiable functions, then so is \((f + g)(x)\) and \((f + g)'(x) = f'(x) + g'(x)\).

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= \lim_{\Delta x \to 0} \frac{f(a + \Delta x) + g(a + \Delta x) - f(a) - g(a)}{\Delta x} \\
= \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(a + \Delta x) - g(a)}{\Delta x}
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Theorem

If $f(x)$ and $g(x)$ are two differentiable functions, then so is $(f + g)(x)$ and $(f + g)'(x) = f'(x) + g'(x)$.

Proof.

\[
(f + g)'(a) = \lim_{\Delta x \to 0} \frac{(f + g)(a + \Delta x) - (f + g)(a)}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{f(a + \Delta x) + g(a + \Delta x) - f(a) - g(a)}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(a + \Delta x) - g(a)}{\Delta x}
\]

\[
= f'(a) + g'(a)
\]
Using the sum rule

Example

Compute the derivative of $f(x) = 8x^{19} - 9\sqrt[4]{x}$. 
Example

Compute the derivative of \( f(x) = 8x^{19} - 9\sqrt[4]{x} \).

\[
f'(x) = \frac{d}{dx}(8x^{19} - 9\sqrt[4]{x})
\]
Using the sum rule

Example

Compute the derivative of \( f(x) = 8x^{19} - 9 \sqrt[4]{x} \).

\[
f'(x) = \frac{d}{dx}(8x^{19} - 9 \sqrt[4]{x})
= \frac{d}{dx}(8x^{19} + (-9) \sqrt[4]{x})
\]
Using the sum rule

Example

Compute the derivative of $f(x) = 8x^{19} - 9 \sqrt[4]{x}$.

$$f'(x) = \frac{d}{dx} (8x^{19} - 9 \sqrt[4]{x})$$

$$= \frac{d}{dx} (8x^{19}) + \frac{d}{dx} (-9x^{\frac{1}{4}})$$
Using the sum rule

Example

Compute the derivative of $f(x) = 8x^{19} - 9\sqrt[4]{x}$.

$$f'(x) = \frac{d}{dx}(8x^{19} - 9\sqrt[4]{x})$$

$$= \frac{d}{dx}(8x^{19}) + \frac{d}{dx}(-9\sqrt[4]{x})$$

$$= 8\frac{d}{dx}(x^{19}) + (-9)\frac{d}{dx}(x^{\frac{1}{4}})$$
Using the sum rule

Example

Compute the derivative of $f(x) = 8x^{19} - 9\sqrt[4]{x}$.

$$f'(x) = \frac{d}{dx}(8x^{19} - 9\sqrt[4]{x})$$

$$= \frac{d}{dx}(8x^{19} + (-9)(4\sqrt{x}))$$

$$= \frac{d}{dx}(8x^{19}) + \frac{d}{dx}(-9x^{1/4})$$

$$= 8\frac{d}{dx}(x^{19}) + (-9)\frac{d}{dx}(x^{1/4})$$

$$= (8)19x^{18} + (-9)\frac{1}{4}x \cdot \frac{-3}{4}$$
Using the sum rule

Example

Compute the derivative of \( f(x) = 8x^{19} - 9\sqrt[4]{x} \).

\[
f'(x) = \frac{d}{dx}(8x^{19} - 9\sqrt[4]{x})
\]

\[
= \frac{d}{dx}(8x^{19} + (-9)\sqrt[4]{x})
\]

\[
= \frac{d}{dx}(8x^{19}) + \frac{d}{dx}(-9x^{\frac{1}{4}})
\]

\[
= 8 \frac{d}{dx}(x^{19}) + (-9) \frac{d}{dx}(x^{\frac{1}{4}})
\]

\[
= (8)19x^{18} + (-9)\frac{1}{4}x^{-\frac{3}{4}}
\]

\[
= 152x^{18} - 9/(4\sqrt[4]{x^3})
\]
If $f(x)$ is a differentiable function with $f(x) > 0$ and $r$ is a real number, then the function $g(x) = (f(x))^r$ is differentiable and $g'(x) = rf(x)^{r-1}f'(x)$. 

**Theorem**

*If $f(x)$ is a differentiable function with $f(x) > 0$ and $r$ is a real number, then the function $g(x) = (f(x))^r$ is differentiable and $g'(x) = rf(x)^{r-1}f'(x)$.*
Example

Compute the derivative of $f(x) = \sqrt[4]{7x^2 + 1}$.
Example

Compute the derivative of $f(x) = \sqrt[4]{7x^2 + 1}$.

$$f'(x) = \frac{d}{dx}((7x^2 + 1)^{\frac{1}{4}})$$
Example

Compute the derivative of $f(x) = \sqrt[4]{7x^2 + 1}$.

$$f'(x) = \frac{d}{dx}((7x^2 + 1)^{\frac{1}{4}})$$

$$= \frac{1}{4}(7x^2 + 1)^{-\frac{3}{4}} \frac{d}{dx}(7x^2 + 1)$$
Computing derivatives using the power rule

Example

Compute the derivative of \( f(x) = \sqrt[4]{7x^2 + 1} \).

\[
f'(x) = \frac{d}{dx} \left((7x^2 + 1)^{\frac{1}{4}}\right)
= \frac{1}{4} (7x^2 + 1)^{-\frac{3}{4}} \frac{d}{dx}(7x^2 + 1)
= \frac{1}{4} (7x^2 + 1)^{-\frac{3}{4}} \left( \frac{d}{dx}(7x^2) + \frac{d}{dx}(1) \right)
\]
Computing derivatives using the power rule

Example

Compute the derivative of \( f(x) = \sqrt[4]{7x^2 + 1} \).

\[
\begin{align*}
f'(x) &= \frac{d}{dx}((7x^2 + 1)^{\frac{1}{4}}) \\
&= \frac{1}{4} (7x^2 + 1)^{-\frac{3}{4}} \frac{d}{dx}(7x^2 + 1) \\
&= \frac{1}{4} (7x^2 + 1)^{-\frac{3}{4}} \left( \frac{d}{dx}(7x^2) + \frac{d}{dx}(1) \right) \\
&= \frac{1}{4} (7x^2 + 1)^{-\frac{3}{4}} \left( 7 \frac{d}{dx}(x^2) + 0 \right)
\end{align*}
\]
Example

Compute the derivative of \( f(x) = \sqrt[4]{7x^2 + 1} \).

\[
f'(x) = \frac{d}{dx} \left( (7x^2 + 1)^{\frac{1}{4}} \right)
= \frac{1}{4} (7x^2 + 1)^{-\frac{3}{4}} \frac{d}{dx} (7x^2 + 1)
= \frac{1}{4} (7x^2 + 1)^{-\frac{3}{4}} \left( \frac{d}{dx} (7x^2) + \frac{d}{dx} (1) \right)
= \frac{1}{4} (7x^2 + 1)^{-\frac{3}{4}} (7 \frac{d}{dx} (x^2) + 0)
= \frac{1}{4} (7x^2 + 1)^{-\frac{3}{4}} ((7)2x)
\]
Computing derivatives using the power rule

Example

Compute the derivative of \( f(x) = \sqrt[4]{7x^2 + 1} \).

\[
f'(x) = \frac{d}{dx}((7x^2 + 1)^{\frac{1}{4}}) \\
= \frac{1}{4}(7x^2 + 1)^{-\frac{3}{4}} \frac{d}{dx}(7x^2 + 1) \\
= \frac{1}{4}(7x^2 + 1)^{-\frac{3}{4}} \left( \frac{d}{dx}(7x^2) + \frac{d}{dx}(1) \right) \\
= \frac{1}{4}(7x^2 + 1)^{-\frac{3}{4}} \left( 14x^2 + 0 \right) \\
= \frac{1}{4}(7x^2 + 1)^{-\frac{3}{4}} \left( 7 \cdot 2x \right) \\
= \frac{7x}{(2\sqrt[4]{(7x^2 + 1)^3})}
\]
Another derivative computation

Example

Compute the derivative of $f(x) = \frac{5}{\sqrt{8x^3 + 2x}}$. 
Example

Compute the derivative of \( f(x) = \frac{5}{\sqrt{8x^3+2x}}. \)

\[
f'(x) = \frac{d}{dx} \left( \frac{5}{\sqrt{8x^3 + 2x}} \right)
\]
Example

Compute the derivative of \( f(x) = \frac{5}{\sqrt{8x^3 + 2x}} \).

\[
f'(x) = \frac{d}{dx} \left( \frac{5}{\sqrt{8x^3 + 2x}} \right)
= 5 \frac{d}{dx} (8x^3 + 2x)^{\frac{1}{2}}
\]
Another derivative computation

Example

Compute the derivative of \( f(x) = \frac{5}{\sqrt{8x^3+2x}} \).

\[
f'(x) = \frac{d}{dx} \left( \frac{5}{\sqrt{8x^3+2x}} \right) \\
= 5 \frac{d}{dx} (8x^3 + 2x)^{\frac{1}{2}} \\
= 5 \left( \frac{1}{2} \right) (8x^3 + 2x)^{-\frac{1}{2}} \frac{d}{dx} (8x^3 + 2x)
\]
Another derivative computation

Example

Compute the derivative of $f(x) = \frac{5}{\sqrt{8x^3+2x}}$.

\[
f'(x) = \frac{d}{dx} \left( \frac{5}{\sqrt{8x^3+2x}} \right)
\]
\[
= 5 \frac{d}{dx} (8x^3 + 2x)^{\frac{1}{2}}
\]
\[
= 5 \left( \frac{1}{2} \right) (8x^3 + 2x)^{-\frac{1}{2}} \frac{d}{dx} (8x^3 + 2x)
\]
\[
= \frac{5}{2} (8x^3 + 2x)^{-\frac{1}{2}} (24x^2 + 2)
\]

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Another derivative computation

Example

Compute the derivative of \( f(x) = \frac{5}{\sqrt{8x^3 + 2x}} \).

\[
f'(x) = \frac{d}{dx} \left( \frac{5}{\sqrt{8x^3 + 2x}} \right)
= 5 \frac{d}{dx} (8x^3 + 2x)^{\frac{1}{2}}
= 5 \left( \frac{1}{2} \right) (8x^3 + 2x)^{-\frac{1}{2}} \frac{d}{dx} (8x^3 + 2x)
= \frac{5}{2} (8x^3 + 2x)^{-\frac{1}{2}} (24x^2 + 2)
= \frac{60x^2 + 5}{\sqrt{8x^3 + 2x}}
\]
While we conventionally write functions with an independent variable $x$ and set the result equal to $y$ (ie $y = f(x)$), it is not necessary to use this choice of variables.
Example

Let $f(t) = 3t^2 + 8$. Compute $f'(t)$.
Derivative computation

Example

Let $f(t) = 3t^2 + 8$. Compute $f'(t)$.

If $g(x) = 3x^2 + 8$, then
Example

Let \( f(t) = 3t^2 + 8 \). Compute \( f'(t) \).

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\]

\[
= \frac{d}{dx}(3x^2) + \frac{d}{dx}(8)
\]
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Let \( f(t) = 3t^2 + 8 \). Compute \( f'(t) \).

If \( g(x) = 3x^2 + 8 \), then

\[
g'(x) = \frac{d}{dx}(3x^2 + 8)
= \frac{d}{dx}(3x^2) + \frac{d}{dx}(8)
= 3 \frac{d}{dx}(x^2) + 0
\]
Derivative computation

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Let $f(t) = 3t^2 + 8$. Compute $f'(t)$.

If $g(x) = 3x^2 + 8$, then

$$g'(x) = \frac{d}{dx} (3x^2 + 8)$$

$$= \frac{d}{dx} (3x^2) + \frac{d}{dx} (8)$$

$$= 3 \frac{d}{dx} (x^2) + 0$$

$$= (3)(2)x$$
Example

Let \( f(t) = 3t^2 + 8 \). Compute \( f'(t) \).

If \( g(x) = 3x^2 + 8 \), then

\[
g'(x) = \frac{d}{dx}(3x^2 + 8) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(8) = 3 \frac{d}{dx}(x^2) + 0 = (3)(2)x = 6x
\]
We have alternate notations for the derivative. If \( y = f(x) \) is a differentiable function, then we may write the derivative of \( f \) as

- \( f'(x) \)
- \( y' \)
- \( \frac{d}{dx} f(x) \)
- \( \frac{dy}{dx} \)
Alternate notation for derivatives

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When the independent variable of our function is something other than $x$, then our notation for the derivative reflects this fact.
Alternate independent variables and derivatives

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**Example**

If $x = s(t)$, then we may write the derivative of $s$ (with respect to $t$) as

- $s'(t)$
- $x'$
- $\frac{d}{dt}s(t)$
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Example

Let \( h(y) = \frac{8}{y^3 + \sqrt{y}} \). Compute \( h'(y) \).
A derivative calculation

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Example

Let \( h(y) = \frac{8}{y^3 + \sqrt{y}} \). Compute \( h'(y) \).

\[
\begin{align*}
h'(y) &= \frac{d}{dy} \left( \frac{8}{y^3 + \sqrt{y}} \right) \\
&= \frac{d}{dy} \left( 8(y^3 + y^{1/2})^{-1} \right)
\end{align*}
\]
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\[
\begin{align*}
h'(y) &= \frac{d}{dy} \left( \frac{8}{y^3 + \sqrt{y}} \right) \\
      &= \frac{d}{dy} (8(y^3 + y^{\frac{1}{2}})^{-1}) \\
      &= (8)(-1)(y^3 + y^{\frac{1}{2}})^{-2} \left( \frac{d}{dy}(y^3) + \frac{d}{dy}(y^{\frac{1}{2}}) \right)
\end{align*}
\]
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Example

Let $h(y) = \frac{8}{y^3 + \sqrt{y}}$. Compute $h'(y)$.

\[
h'(y) = \frac{d}{dy} \left( \frac{8}{y^3 + \sqrt{y}} \right)
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\[
\begin{align*}
    h'(y) & = \frac{d}{dy} \left( \frac{8}{y^3 + \sqrt{y}} \right) \\
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    & = (8)(-1)(y^3 + y^{\frac{1}{2}})^{-2} \left( \frac{d}{dy}(y^3) + \frac{d}{dy}(y^{\frac{1}{2}}) \right) \\
    & = -8(y^3 + y^{\frac{1}{2}})^{-2} \left( 3y^2 + \frac{1}{2}y^{-\frac{1}{2}} \right) \\
    & = \frac{-8(3y^2 + \frac{1}{2\sqrt{y}})}{(y^3 + \sqrt{y})^2}
\end{align*}
\]
For now, the notation $\frac{d}{dt}$, $\frac{d}{dx}$, $\frac{d}{dy}$, *et cetera* is used merely to indicate that the independent variable is $t$, $x$, $y$, respectively. When we deal with functions of several variables (in Math 16b), then this notation will carry the additional meaning that we regard all other variables as constant.
Example

Let $a$, $b$, and $c$ be three real numbers. Let $f(t) = at^2 + bt + c$. Compute $f'(t)$.
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$$= \frac{d}{dt}(at^2) + \frac{d}{dt}(bt) + \frac{d}{dt}(c)$$
Example

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\[
\begin{align*}
  f'(t) &= \frac{d}{dt}(at^2 + bt + c) \\
        &= \frac{d}{dt}(at^2) + \frac{d}{dt}(bt) + \frac{d}{dt}(c) \\
        &= a\frac{d}{dt}(t^2) + b\frac{d}{dt}(t) + 0
\end{align*}
\]
Example

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$$= a(2)t + b(1) + 0$$
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Let $a$, $b$, and $c$ be three real numbers. Let $f(t) = at^2 + bt + c$. Compute $f'(t)$.

\[
f'(t) = \frac{d}{dt} (at^2 + bt + c)
\]

\[
= \frac{d}{dt} (at^2) + \frac{d}{dt} (bt) + \frac{d}{dt} (c)
\]

\[
= a \frac{d}{dt} (t^2) + b \frac{d}{dt} (t) + 0
\]

\[
= a(2)t + b(1) + 0
\]

\[
= 2at + b
\]
The second derivative

Definition

If $f(x)$ is a differentiable function of $x$, then $f'(x)$ is another function of $x$. The derivative of $f'(x)$, denoted $f''(x)$, is called the second derivative of $f$. We call $f'(x)$ the first derivative of $f$. 
Alternate notations for second derivatives

As with the derivative itself, there are alternate notations for the second derivative. If \( y = f(x) \), then we may denote the second derivative of \( f \) by:

- \( f''(x) \)
- \( y'' \)
- \( f^{(2)}(x) \)
- \( \frac{d^2}{dx^2} f(x) \)
- \( \frac{d^2 y}{dx^2} \)
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- \( \frac{d^2}{dx^2} f(x) \)
- \( \frac{d^2 y}{dx^2} \)
Computing a second derivative

Example

Let $f(t) = 4t + \frac{8}{\sqrt[3]{t}}$. Compute $f''(t)$. 
Computing a second derivative

Example

Let $f(t) = 4t + \frac{8}{3\sqrt{t}}$. Compute $f''(t)$.

We start by computing $f'(t)$. 
Computing a second derivative

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Let \( f(t) = 4t + \frac{8}{\sqrt[3]{t}} \). Compute \( f''(t) \).

We start by computing \( f'(t) \).

\[
f'(t) = \frac{d}{dt} \left(4t + \frac{8}{\sqrt[3]{t}}\right)
\]
Computing a second derivative

Example

Let $f(t) = 4t + \frac{8}{\sqrt[3]{t}}$. Compute $f''(t)$.

We start by computing $f'(t)$.

$$f'(t) = \frac{d}{dt}(4t + \frac{8}{\sqrt[3]{t}})$$

$$= \frac{d}{dt}(4t) + \frac{d}{dt}(8t^{-\frac{1}{3}})$$
Computing a second derivative

**Example**

Let \( f(t) = 4t + \frac{8}{\sqrt[3]{t}} \). Compute \( f''(t) \).

We start by computing \( f'(t) \).

\[
f'(t) = \frac{d}{dt}(4t + \frac{8}{\sqrt[3]{t}})
\]

\[
= \frac{d}{dt}(4t) + \frac{d}{dt}(8t^{-\frac{1}{3}})
\]

\[
= 4 + 8\left(-\frac{1}{3}\right)t^{-\frac{4}{3}}
\]
Example

Let \( f(t) = 4t + \frac{8}{3\sqrt[3]{t}} \). Compute \( f''(t) \).

We start by computing \( f'(t) \).

\[
\begin{align*}
f'(t) &= \frac{d}{dt} \left( 4t + \frac{8}{3\sqrt[3]{t}} \right) \\
&= \frac{d}{dt} \left( 4t \right) + \frac{d}{dt} \left( 8t^{-\frac{1}{3}} \right) \\
&= 4 + 8 \left( -\frac{1}{3} \right) t^{-\frac{4}{3}} \\
&= 4 - \frac{8}{3^{\frac{3}{2}}t^{\frac{4}{3}}} \\
&= 4 - \frac{8}{3^{\frac{3}{2}}t^{4}}
\end{align*}
\]
Computation continued

\[ f''(t) = \frac{d}{dt} f'(t) \]
\[ f''(t) = \frac{d}{dt} f'(t) = \frac{d}{dt} \left( 4 - \frac{8}{3 \sqrt{t^4}} \right) \]
Computation continued

\[ f''(t) = \frac{d}{dt} f'(t) \]
\[ = \frac{d}{dt} \left( 4 - \frac{8}{3\sqrt[3]{t^4}} \right) \]
\[ = \frac{d}{dt} (4) - \frac{8}{3} \frac{d}{dt} (t^{-4/3}) \]
Computation continued

\[ f''(t) = \frac{d}{dt} f'(t) \]

\[ = \frac{d}{dt} \left( 4 - \frac{8}{3\sqrt[3]{t^4}} \right) \]

\[ = \frac{d}{dt} (4) - \frac{8}{3} \frac{d}{dt} (t^{-\frac{4}{3}}) \]

\[ = 0 - \frac{8}{3} \frac{-4t^{-\frac{4}{3}}}{3} \]

\[ = 0 - \frac{8}{3} \frac{-4}{3} t^{-\frac{7}{3}} \]
Computation continued

\[ f''(t) = \frac{d}{dt} f'(t) \]
\[ = \frac{d}{dt} \left( 4 - \frac{8}{3\sqrt[3]{t^4}} \right) \]
\[ = \frac{d}{dt} (4) - \frac{8}{3} \frac{d}{dt} \left( t^{-\frac{4}{3}} \right) \]
\[ = 0 - \frac{8}{3} \frac{-4}{3} t^{-\frac{7}{3}} \]
\[ = \frac{32}{9\sqrt[3]{t^7}} \]
Another computation of a second derivative

Example

Let \( x = f(y) = \frac{1}{y} \). Compute \( \frac{d^2x}{dy^2} \).
Another computation of a second derivative

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Let \( x = f(y) = \frac{1}{y} \). Compute \( \frac{d^2 x}{dy^2} \).

\[
\frac{d^2 x}{dy^2} = \frac{d}{dy} \left( \frac{dx}{dy} \right)
\]
Another computation of a second derivative

Example

Let \( x = f(y) = \frac{1}{y} \). Compute \( \frac{d^2 x}{dy^2} \).

\[
\frac{d^2 x}{dy^2} = \frac{d}{dy} \left( \frac{dx}{dy} \right) \\
= \frac{d}{dy} \left( \frac{1}{y} \right)
\]
Another computation of a second derivative

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Let \( x = f(y) = \frac{1}{y} \). Compute \( \frac{d^2 x}{dy^2} \).

\[
\frac{d^2 x}{dy^2} = \frac{d}{dy} \left( \frac{dx}{dy} \right) \\
= \frac{d}{dy} \left( \frac{d}{dy} \left( \frac{1}{y} \right) \right) \\
= \frac{d}{dy} \left( \frac{d}{dy} \left( y^{-1} \right) \right)
\]
Another computation of a second derivative

Example

Let \( x = f(y) = \frac{1}{y} \). Compute \( \frac{d^2 x}{dy^2} \).

\[
\frac{d^2 x}{dy^2} = \frac{d}{dy} \left( \frac{dx}{dy} \right) \\
= \frac{d}{dy} \left( \frac{d}{dy} \left( \frac{1}{y} \right) \right) \\
= \frac{d}{dy} \left( -y^{-1} \right) \\
= \frac{d}{dy} (-y^{-2})
\]
Another computation of a second derivative

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Let \( x = f(y) = \frac{1}{y} \). Compute \( \frac{d^2 x}{dy^2} \).

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\frac{d^2 x}{dy^2} = \frac{d}{dy} \left( \frac{dx}{dy} \right)
= \frac{d}{dy} \left( \frac{d}{dy} \left( \frac{1}{y} \right) \right)
= \frac{d}{dy} \left( \frac{d}{dy} \left( y^{-1} \right) \right)
= \frac{d}{dy} \left( -y^{-2} \right)
= 2y^{-3}
\]
If $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ is a function of $x$. If $a$ is a real number in the domain of $f'(x)$, then we write $\frac{dy}{dx}\big|_{x=a}$ for $f'(a)$, the result of evaluating the derivative of $f$ at $a$. 
If \( y = f(x) \), then \( \frac{dy}{dx} = f'(x) \) is a function of \( x \). If \( a \) is a real number in the domain of \( f'(x) \), then we write \( \left. \frac{dy}{dx} \right|_{x=a} \) for \( f'(a) \), the result of evaluating the derivative of \( f \) at \( a \). Likewise, we write \( \left. \frac{d^2y}{dx^2} \right|_{x=a} \) for \( f''(a) \).
Evaluating a second derivative

Example

Let \( y = g(x) = x^2 - \sqrt{x} \). Compute \( \frac{d^2}{dx^2} g \bigg|_{x=4} \).
Evaluating a second derivative

Example
Let \( y = g(x) = x^2 - \sqrt{x} \). Compute \( \frac{d^2}{dx^2} g \big|_{x=4} \).

By definition, \( \frac{d^2}{dx^2} g \big|_{x=4} = g''(4) \). Thus, we need only compute \( g''(x) \) and then evaluate at 4.
Computation continued

\[ g''(x) = \frac{d^2}{dx^2}(x^2 - \sqrt{x}) \]
Computation continued

\[ g''(x) = \frac{d^2}{dx^2}(x^2 - \sqrt{x}) \]

\[ = \frac{d}{dx} \left( \frac{d}{dx}(x^2 - x^{1/2}) \right) \]
Computation continued

\[
g''(x) = \frac{d^2}{dx^2}(x^2 - \sqrt{x})
\]

\[
= \frac{d}{dx}\left( \frac{d}{dx}(x^2 - x^{\frac{1}{2}}) \right)
\]

\[
= \frac{d}{dx}\left(2x - \frac{1}{2}x^{-\frac{1}{2}} \right)
\]
\[ g''(x) = \frac{d^2}{dx^2} (x^2 - \sqrt{x}) \]
\[ = \frac{d}{dx} \left( \frac{d}{dx} (x^2 - x^{\frac{1}{2}}) \right) \]
\[ = \frac{d}{dx} (2x - \frac{1}{2}x^{-\frac{1}{2}}) \]
\[ = 2 - \frac{1}{2} \left( -\frac{1}{2} \right) x^{-\frac{3}{2}} \]
Computation continued

\[ g''(x) = \frac{d^2}{dx^2}(x^2 - \sqrt{x}) \]
\[ = \frac{d}{dx}\left(\frac{d}{dx}(x^2 - x^{1/2})\right) \]
\[ = \frac{d}{dx}(2x - \frac{1}{2}x^{-1/2}) \]
\[ = 2 - \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2} \]
\[ = 2 + \frac{1}{4\sqrt{x^3}} \]
Evaluating at $x = 4$, we conclude
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$$\frac{d^2}{dx^2} g|_{x=4} = g''(4)$$
Evaluating at $x = 4$, we conclude

\[ \frac{d^2}{dx^2} g \bigg|_{x=4} = g''(4) \]

\[ = 2 + \frac{1}{4\sqrt{4^3}} \]
Evaluating at $x = 4$, we conclude

$$\frac{d^2}{dx^2} g|_{x=4} = g''(4)$$

$$= 2 + \frac{1}{4\sqrt{4^3}}$$

$$= 2 + \frac{1}{4(8)}$$
Evaluating at $x = 4$, we conclude

$$\left. \frac{d^2}{dx^2} g \right|_{x=4} = g''(4)$$

$$= 2 + \frac{1}{4\sqrt{4^3}}$$

$$= 2 + \frac{1}{4(8)}$$

$$= 2 + \frac{1}{32}$$
Evaluating at $x = 4$, we conclude

$$\frac{d^2}{dx^2} g \big|_{x=4} = g''(4)$$

$$= 2 + \frac{1}{4\sqrt{4^3}}$$

$$= 2 + \frac{1}{4(8)}$$

$$= 2 + \frac{1}{32}$$

$$= \frac{65}{32}$$
Rates of change

We can interpret the derivative of a function as a rate of change.
We can interpret the derivative of a function as a rate of change. If $x_{\text{initial}}$ is some initial value of $x$ and $x_{\text{final}}$ is the final value of $x$, then we can regard $\Delta x := x_{\text{final}} - x_{\text{initial}}$ as a change in $x$. If $y = f(x)$ is a function with $x_{\text{initial}}$ and $x_{\text{final}}$ in the domain of $f$, ...
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$$\Delta y := f(x_{\text{final}}) - f(x_{\text{initial}})$$
Rates of change

We can interpret the derivative of a function as a rate of change. If $x_{\text{initial}}$ is some initial value of $x$ and $x_{\text{final}}$ is the final value of $x$, then we can regard $\Delta x := x_{\text{final}} - x_{\text{initial}}$ as a change in $x$. If $y = f(x)$ is a function with $x_{\text{initial}}$ and $x_{\text{final}}$ in the domain of $f$, then

$$\Delta y := f(x_{\text{final}}) - f(x_{\text{initial}}) = f(x_{\text{initial}} + \Delta x) - f(x_{\text{initial}})$$
Rates of change

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$$\Delta y := f(x_{\text{final}}) - f(x_{\text{initial}})$$

$$= f(x_{\text{initial}} + \Delta x) - f(x_{\text{initial}})$$

is the change in the value of the function from the initial point to the final point.
Rates of change

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$$\Delta y := f(x_{\text{final}}) - f(x_{\text{initial}})$$

$$= f(x_{\text{initial}} + \Delta x) - f(x_{\text{initial}})$$

is the change in the value of the function from the initial point to the final point.

The quotient $\frac{\Delta y}{\Delta x}$ may be regarded as the average rate of change of $f(x)$ from $x_{\text{initial}}$ to $x_{\text{final}} = x_{\text{initial}} + \Delta x$. 
Example

Let $f(x) = x^3 + 1$. Compute the average rate of change from $x = 0.5$ to $x = 0.6$. 
Example

Let $f(x) = x^3 + 1$. Compute the average rate of change from $x = 0.5$ to $x = 0.6$.

We compute $f(0.6) = (0.6)^3 + 1 = 1.216$
A computation of an average rate of change

Example

Let \( f(x) = x^3 + 1 \). Compute the average rate of change from \( x = 0.5 \) to \( x = 0.6 \).

We compute \( f(0.6) = (0.6)^3 + 1 = 1.216 \) and \( f(0.5) = (0.5)^3 + 1 = 1.125 \).
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Our expression for the average rate of change of a function is identical to our formula for the slope of the secant line between \((x_{\text{initial}}, f(x_{\text{initial}}))\) and \((x_{\text{final}}, f(x_{\text{final}}))\).
Another expression for the average rate of change

If we set \( a := x_{\text{initial}} \) and \( h := \Delta x \), so that \( x_{\text{final}} = a + h \), then the average rate of change of the function \( y = f(x) \) from \( a \) to \( a + h \) is

\[
\frac{f(a + h) - f(a)}{h}
\]
Derivative as instantaneous rate of change

If $f(x)$ is differentiable at $a$, then

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

may be regarded as the instantaneous rate of change of $f$ at $a$. 
Computing an instantaneous rate of change

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Let $f(x) = x^3 + 1$. Compute the instantaneous rate of change of $f$ at $a = 0.5$. 
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We compute $f'(x) = 3x^2$. 
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Let \( f(x) = x^3 + 1 \). Compute the instantaneous rate of change of \( f \) at \( a = 0.5 \).

We compute \( f'(x) = 3x^2 \). Thus, the instantaneous rate of change of \( f(x) \) at \( a = 0.5 \) is \( 3(0.5)^2 = 0.75 \).
Velocity as a derivative

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**Definition**

The velocity of an object is the instantaneous rate of change of its position. That is, if its position is given by $x(t)$, then its velocity is $v(t) = x'(t)$. 
Warning on speed and velocity

Warning

The speed of an object is the magnitude of its velocity. An object which is moving backwards has a negative velocity, but its speed can never be negative.
Computing velocity

Example

An object dropped from a height one thousand feet in a vacuum has a position of $x(t) = 1000 - 16t^2$ after $t$ seconds. What is its velocity after ten seconds?
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An object dropped from a height one thousand feet in a vacuum has a position of \( x(t) = 1000 - 16t^2 \) after \( t \) seconds. What is its velocity after ten seconds?

\[ v(t) = x'(t) = -32t. \]

Thus, the velocity of the object after ten seconds is \(-320\) feet per second.
Definition

The rate of change of the velocity of an object is its acceleration. That is, if \( v(t) \) denotes the velocity of the object at time \( t \), then its acceleration is \( a(t) = v'(t) \).
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The velocity of an object having position \( x(t) \) is \( v(t) = x'(t) \). Thus, the acceleration of that object is \( a(t) = v'(t) = x''(t) \).
Computing acceleration

Example

In the above example of the object dropped in a vacuum, compute its acceleration after ten seconds.
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We already know that the velocity is \( v(t) = -32t \). Thus, the acceleration is \( a(t) = v'(t) = -32 \). Therefore, after ten seconds the acceleration of \(-32\) feet per second per second.
By definition,

\[ \text{[rate of change]} = \frac{\Delta y}{\Delta x} \]
∆y in terms of ∆x

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Multiplying by ∆x, we obtain

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Multiplying by \(\Delta x\), we obtain

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The derivative is well-approximated by the the slopes of the secant lines. Thus, we have

\[
\frac{dy}{dx}\Delta x \approx \Delta y
\]
Approximating a change using derivative

Example
Let \( p(t) \) denote the number of pages read after \( t \) minutes. If \( p(60) = 50 \) and \( p'(60) = 0.5 \), estimate the number of pages read after one hour and five minutes.
Approximating a change using derivative

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Let \( p(t) \) denote the number of pages read after \( t \) minutes. If \( p(60) = 50 \) and \( p'(60) = .5 \), estimate the number of pages read after one hour and five minutes.

\[ p'(t) \] is the rate at which the pages are read.
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Example

Let $p(t)$ denote the number of pages read after $t$ minutes. If $p(60) = 50$ and $p'(60) = .5$, estimate the number of pages read after one hour and five minutes.

$p'(t)$ is the rate at which the pages are read. We estimate $p(65) \approx p(60) + 5p'(60) = 50 + 5(0.5) = 52.5$. 
In economics, derivatives are often introduced through the term \textit{marginal}. For example, if $R(x)$ is the revenue generated by the application of $x$ units of capital, then the \textit{marginal revenue} is the additional revenue generated by an additional unit of capital.
Computing marginal revenue

Example
Suppose that the profit from a certain enterprise as a function of the number $x$ of worker-hours of labor is given by $P(x) = 100(x + \frac{1}{x})$. What is the marginal profit at 8 worker-hours?
Computing marginal revenue

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Suppose that the profit from a certain enterprise as a function of the number $x$ of worker-hours of labor is given by $P(x) = 100(x + \frac{1}{x})$. What is the marginal profit at 8 worker-hours?

$P'(x) = 100(1 - x^{-2})$
Computing marginal revenue

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Suppose that the profit from a certain enterprise as a function of the number $x$ of worker-hours of labor is given by $P(x) = 100(x + \frac{1}{x})$. What is the marginal profit at 8 worker-hours?

$$P'(x) = 100(1 - x^{-2})$$

Thus, $P'(8) = 100(1 - \frac{1}{64}) = \frac{6300}{64}$. 