Math 16A (Autumn 2005)

Thomas Scanlon

University of California, Berkeley

Week 15
Mean value

Definition

If $y_1, \ldots, y_n$ is a finite sequence of real numbers, then the average or mean value of the sequence is

$$\mu = \frac{y_1 + \cdots + c_n}{n}$$
Average value of a continuous function

If $f(x)$ is a continuous function defined on some interval $[a, b]$, we can find the average value of $f(x)$ from the notion of the average of a finite sequence. Fix a number $n$ and sample consider the sequence $f(x_1), \ldots, f(x_n)$ where $a \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq b$ is a sequence of points between $[a, b]$ chosen with a “uniform” distribution. To make the notion of “uniform” precise, we could break $[a, b]$ into $n$ intervals each of length $\Delta x = \frac{b - a}{n}$ and choose $x_i$ in the $i^{th}$ interval.
The average of the sequence is then

\[ \mu = \frac{f(x_1) + \cdots + f(x_n)}{n} \]

\[ = \frac{1}{b-a}(f(x_1) + \cdots + f(x_n)) \frac{1}{n} \]

\[ = \frac{1}{b-a}(f(x_1) + \cdots + f(x_n))\Delta x \]

This last expression is \( \frac{1}{b-a} \) [Riemann sum of \( f(x) \) with respect to \( n \) and \( x_1, \ldots, x_n \)].
As the Riemann sums approach the integral $\int_{a}^{b} f(x)dx$, we may define the average value of a continuous function over an interval in terms of its integral.

**Definition**

Let $f(x)$ be a continuous function on the interval $[a, b]$ we define the average value or mean of $f(x)$ over $[a, b]$ to be

$$
\mu = \frac{1}{b - a} \int_{a}^{b} f(x)dx
$$
Average value in an example

Example
Compute the average value of \( f(x) = e^x - x \) over the interval \([1, 3]\).
A solution

\[
\mu = \frac{1}{3 - 1} \int_1^3 (e^x - x)dx \\
= \frac{1}{2} [e^x - \frac{1}{2}x^2]|_1^3 \\
= \frac{1}{2} [(e^2 - 2) - (e - \frac{1}{2})] \\
= \frac{1}{2} e^2 - \frac{1}{2} e - \frac{3}{4}
\]
Continuous income stream

If a single payment of $A$ dollars is made $t$ years from now and we assume a fixed inflation rate of $r$, then that payment is worth $A e^{-rt}$ dollars in present value.

How can we correctly appraise the value of a continuous stream of income over time?

**Example**

Suppose that we expect to receive payments at a constant rate of $A$ dollars per year. After $b$ years we will have received $Ab$ dollars. How much are these payments worth in present dollars?
A solution

We can approximate the continuous stream of income as a sequence of discrete payments spread evenly over time. Fix a number \( n \), let \( \Delta t = \frac{b}{n} \) and let \( t_i = i\Delta t \) for \( i = 1, \ldots, n \). If we aggregate the payments over each of these intervals, then we have a payment of size \( A\Delta t \) at time \( t_1 \), one of size \( A\Delta t \) at time \( t_2 \), and so on.
The payment of $A\Delta t$ at time $t_1$ is worth $(A\Delta t)e^{-rt_1}$ in present dollars.
The payment at time $t_2$ is worth $(A\Delta t)e^{-rt_2}$ in present dollars, and so on.
Thus, the present value of this constant income stream is approximately

$$Ae^{-rt_1}\Delta t + Ae^{-rt_2}\Delta t + \cdots + Ae^{-rt_n}\Delta t$$

We recognize this expression as a Riemann sum for $f(t) = Ae^{-rt}$.
So, the present value of the constant income stream is

$$\int_0^b Ae^{-rt} \, dt = \left[ \frac{-A}{r} e^{-rt} \right]_0^b = \frac{A}{r} (1 - e^{-rb})$$
Let $f(x)$ be a continuous, non-negative function defined on some interval $[a, b]$. If the region under the graph of $y = f(x)$ is spun around the $x$-axis to make a solid, then this object has volume

$$\int_{a}^{b} \pi (f(x))^2 \, dx$$
Volume of a cone

Example

Find the volume of the cone with height 3 and slant height 5.
A solution

Such a cone may be obtained by sweeping the line segment from (0, 0) to (3, b) around the x-axis where b is chosen so that the segment has length 5. By the Pythagorean theorem we need $b^2 + 9 = 25$ so that $b = 4$. The line segment is the graph of $f(x) = \frac{4}{3}x$ on $[0, 3]$. So, the volume of the cone is

$$\int_{0}^{3} \pi \frac{16}{9} x^2 \, dx = 16\pi$$