6.5: Applications of the definite integral

Math 16A (Autumn 2005)

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Week 15
Mean value

Definition

If $y_1, \ldots, y_n$ is a finite sequence of real numbers, then the average or mean value of the sequence is

$$\mu = \frac{y_1 + \cdots + c_n}{n}$$
Average value of a continuous function

If \( f(x) \) is a continuous function defined on some interval \([a, b]\), we can find the **average value** of \( f(x) \) from the notion of the average of a finite sequence.
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Fix a number \( n \) and sample consider the sequence \( f(x_1), \ldots, f(x_n) \)
where \( a \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq b \) is a sequence of points between \([a, b]\) chosen with a “uniform” distribution.
Average value of a continuous function

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To make the notion of “uniform” precise, we could break $[a, b]$ into $n$ intervals each of length $\Delta x = \frac{b - a}{n}$ and choose $x_i$ in the $i^{th}$ interval.
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$$\mu = \frac{f(x_1) + \cdots + f(x_n)}{n}$$
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\[ \mu = \frac{f(x_1) + \cdots + f(x_n)}{n} = \frac{1}{n} \left( f(x_1) + \cdots + f(x_n) \right) \]
The average of the sequence is then

\[
\mu = \frac{f(x_1) + \cdots + f(x_n)}{n} = \frac{1}{b-a} (f(x_1) + \cdots + f(x_n)) \frac{b-a}{n}
\]
Average value, continued

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\[ \mu = \frac{f(x_1) + \cdots + f(x_n)}{n} \]

\[ = \frac{1}{n} (f(x_1) + \cdots + f(x_n)) \frac{b - a}{n} \]

\[ = \frac{1}{b - a} (f(x_1) + \cdots + f(x_n)) \Delta x \]
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This last expression is \( \frac{1}{b - a} \) [Riemann sum of \( f(x) \) with respect to \( n \) and \( x_1, \ldots, x_n \)].
As the Riemann sums approach the integral $\int_a^b f(x) \, dx$, we may define the average value of a continuous function over an interval in terms of its integral.
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**Definition**

Let $f(x)$ be a continuous function on the interval $[a, b]$ we define the average value or mean of $f(x)$ over $[a, b]$ to be

$$\mu = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx$$
Average value in an example

Example

Compute the average value of \( f(x) = e^x - x \) over the interval \([1, 3]\).
A solution

\[
\mu = \frac{1}{3-1} \int_1^3 (e^x - x) \, dx
\]
A solution

\[
\mu = \frac{1}{3-1} \int_1^3 (e^x - x) \, dx \\
= \left. \frac{1}{2} [e^x - \frac{1}{2} x^2] \right|_1^2
\]
6.5: Applications of the definite integral

A solution

\[ \mu = \frac{1}{3 - 1} \int_{1}^{3} (e^{x} - x) \, dx \]

\[ = \frac{1}{2} [e^{x} - \frac{1}{2} x^{2}] \bigg|_{1}^{3} \]

\[ = \frac{1}{2} [(e^{2} - 2) - (e - \frac{1}{2})] \]
A solution

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\]

\[
= \frac{1}{2} \left[ (e^2 - 2) - (e - \frac{1}{2}) \right]
\]

\[
= \frac{1}{2} e^2 - \frac{1}{2} e - \frac{3}{4}
\]
If a single payment of $A$ dollars is made $t$ years from now and we assume a fixed inflation rate of $r$, then that payment is worth $Ae^{-rt}$ dollars in present value.
Continuous income stream

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Example
Suppose that we expect to receive payments at a constant rate of $A$ dollars per year. After $b$ years we will have received $Ab$ dollars. How much are these payments worth in present dollars?
We can approximate the continuous stream of income as a sequence of discrete payments spread evenly over time.
A solution

We can approximate the continuous stream of income as a sequence of discrete payments spread evenly over time. Fix a number $n$, let $\Delta t = \frac{b}{n}$ and let $t_i = i\Delta t$ for $i = 1, \ldots, n$. 
We can approximate the continuous stream of income as a sequence of discrete payments spread evenly over time. Fix a number \( n \), let \( \Delta t = \frac{b}{n} \) and let \( t_i = i\Delta t \) for \( i = 1, \ldots, n \). If we aggregate the payments over each of these intervals, then we have a payment of size \( A\Delta t \) at time \( t_1 \), one of size \( A\Delta t \) at time \( t_2 \), and so on.
The payment of $A\Delta t$ at time $t_1$ is worth $(A\Delta t)e^{-rt_1}$ in present dollars.
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Thus, the present value of this constant income stream is approximately

$$Ae^{-rt_1}\Delta t + Ae^{-rt_2}\Delta t + \cdots + Ae^{-rt_n}\Delta t$$
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We recognize this expression as a Riemann sum for $f(t) = Ae^{-rt}$.
So, the present value of the constant income stream is

$$\int_{0}^{b} Ae^{-rt} \, dt = \left[ -\frac{A}{r} e^{-rt} \right]_{0}^{b}$$

$$= \frac{A}{r} (1 - e^{-rb})$$
Volumes of revolved solids

Let $f(x)$ be a continuous, non-negative function defined on some interval $[a, b]$. If the region under the graph of $y = f(x)$ is spun around the $x$-axis to make a solid, then this object has volume

$$\int_a^b \pi (f(x))^2 \, dx$$
Volume of a cone

Example

Find the volume of the cone with height 3 and slant height 5.
A solution

Such a cone may be obtained by sweeping the line segment from $(0, 0)$ to $(3, b)$ around the $x$-axis where $b$ is chosen so that the segment has length 5.
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The line segment is the graph of $f(x) = \frac{4}{3}x$ on $[0, 3]$. 
So, the volume of the cone is 

$$
\int_{0}^{3} \pi \frac{16}{9} x^2 \, dx = 16\pi
$$