Math 16A (Autumn 2005)

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Week 14
Our method of computing distances from the velocity is an instance of the fundamental theorem of calculus.
Theorem

If \( f(x) \) is a continuous function with domain \([a, b]\) and

\[
\int f(x) \, dx = F(x) + C
\]

then,

\[
F(b) - F(a) = \int_a^b f(x) \, dx
\]
We sometimes write

\[ F(x)\bigg|^{b}_{a} = F(b) - F(a) \]

or

\[ F(x)\bigg|_{x=a}^{x=b} = F(b) - F(a) \]

So that if

\[ \int f(x)\,dx = F(x) + C \]

then

\[ \int_{a}^{b} f(x)\,dx = F(x)\bigg|^{b}_{a} \]
Using the fundamental theorem in an example

Example
Compute the area under the graph of $f(x) = x^2$ between $a = 1$ and $b = 3$. 
A solution

We know

\[ \int x^2 \, dx = \frac{1}{3} x^3 + C \]

So,

\[ \int_1^3 x^2 \, dx = \frac{1}{3} 3^3 - \frac{1}{3} 1^3 \]

\[ = \frac{28}{3} \]

\[ = 9 \frac{1}{3} \]
Example

Approximate $\int_{1}^{3} x^2 \, dx$ by the Riemann sum using $n = 2$ subdivisions and the midpoints of the intervals as sample points.
A solution

- $n = 2$
- $a = 1$
- $b = 3$
- $\Delta x = \frac{b-a}{n} = 1$
- $x_1 = \frac{3}{2}$ and
- $x_2 = \frac{5}{2}$

So that the Riemann sum is $(\frac{3}{2})^2 \cdot 1 + (\frac{5}{2})^2 \cdot 1 = (\frac{34}{4}) = 9 \frac{1}{2}$. This result differs from the true area by $\frac{1}{6}$. 

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Another computation in an example

Example

Compute

\[ \int_1^4 (3e^{4x} - \sqrt{x} + 2x) \, dx \]
\[
\int (3e^{4x} - \sqrt{x} + 2x) \, dx = 3 \int e^{4x} \, dx - \int x^{\frac{1}{2}} \, dx + 2 \int x^{1} \, dx
\]
\[
= \frac{3}{4} e^{4x} - \frac{2}{3} x^{\frac{3}{2}} + x^{2} + C
\]

Using the fundamental theorem,

\[
\int_{1}^{4} (3e^{4x} - \sqrt{x} + 2x) \, dx = \left( \frac{3}{4} e^{4x} - \frac{2}{3} x^{\frac{3}{2}} + x^{2} \right)_{1}^{4}
\]
\[
= \left( \frac{3}{4} e^{16} - \frac{2}{3} (4)^{\frac{3}{2}} + 16 \right) - \left( \frac{3}{4} e^{4} - \frac{2}{3} + 1 \right)
\]
\[
= \frac{3}{4} (e^{16} - e^{4}) + \frac{31}{3}
\]
Another example

Example

Compute

\[
\int_{4}^{9} \frac{7}{x - 3} \, dx
\]
A solution

Differentiating, we see that for \( x > 3 \)

\[
\int \frac{1}{x-3} \, dx = \ln(x-3) + C
\]

Hence,

\[
\int_{4}^{9} \frac{7}{x-3} \, dx = 7 \ln(x-3)\big|_{4}^{9} \\
= 7 \ln(6) - 7 \ln(1) \\
= 7 \ln(6)
\]
While we introduced the definite integral to compute the area bounded by the graph of a positive function and the $x$-axis, it can be used to compute the areas of more complicated regions in the $xy$-plane.
Areas between curves

**Fact**

If \( f(x) \) and \( g(x) \) are two continuous functions defined on the interval \([a, b]\) and \( f(x) \geq g(x) \) for all \( x \), then the area between the curves \( y = f(x) \) and \( y = g(x) \) is

\[
\int_{a}^{b} [f(x) - g(x)] \, dx
\]
Example

Find the area between the curves $f(x) = x^3$ and $g(x) = e^{-x}$ defined on the interval $[1, 2]$. 
A solution

\[ \int_{1}^{2} (x^3 - e^{-x}) \, dx = \left[ \frac{1}{4}x^4 + e^{-x} \right]_{1}^{2} \]

\[ = (4 + e^{-2}) - \left( \frac{1}{4} + e^{-1} \right) \]

\[ = 3 \frac{3}{4} + e^{-2} - e^{-1} \]
If the sign of \( f(x) - g(x) \) changes, then to find the area between the curves \( y = f(x) \) and \( y = g(x) \), one must break the interval into the region on which \( f(x) \geq g(x) \) and the region on which \( f(x) \leq g(x) \). The area is then the difference between the integral of \( f(x) - g(x) \) over the region where \( f(x) \geq g(x) \) and the integral of \( g(x) - f(x) \) over the region where \( g(x) \geq f(x) \).
Example

Find the area between $y = f(x) = x^2 - x + 2$ and $y = g(x) = 2x$ over the interval $[0, 3]$. 
A solution

We see that \( f(x) - g(x) = x^2 - 3x + 2 = (x - 1)(x - 2) \) is greater than or equal to zero for \( 0 \leq x \leq 1 \) and \( 2 \leq x \leq 3 \). It is less than or equal to zero for \( 1 \leq x \leq 2 \).

Thus, the area between the curves is given by

\[
\int_0^1 (x^2 - x + 2)\,dx + \int_1^2 -(x^2 - x + 2)\,dx + \int_2^3 (x^2 - x + 2)\,dx
\]

We compute

\[
\int (x^2 - x + 2)\,dx = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + C
\]

Write \( F(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x \).

The area is

\[
[F(x)]_0^1 - [F(x)]_1^2 + [F(x)]_2^3 = \frac{5}{6} + \frac{1}{6} + \frac{5}{6} = \frac{11}{6}
\]