Relative rate of change

Definition

For a differentiable function $f(t)$, the relative rate of change of $f$ at $t$ is the quantity

$$\frac{f'(t)}{f(t)}$$

The relative rate of change of $f$ expressed as a percentage is called the percentage rate of change.
Recall that for a function $f(t)$ with $f(t) > 0$, we have

$$\frac{d}{dt} \ln(f(t)) = \frac{f'(t)}{f(t)}$$
Relative rates of change in economics

The percentage rate of change is used to describe inflation of prices, economic growth rates, and sales growth rates.
Example

If $P(t)$ gives the price for some commodity at time $t$ years and we assume that $P(t)$ is a differentiable function, then the inflation rate for that commodity is the percentage rate of change of $P$. Suppose that the price is given by $P(t) = 5e^{2\sqrt[3]{t}}$. What is the inflation rate for this commodity after eight years?
A solution

We compute

\[ P'(t) = 5e^{2\sqrt[3]{t}}2\left(\frac{1}{3}\right)t^{-\frac{2}{3}} \]

Evaluating at \( t = 8 \), we obtain

\[ P'(8) = \frac{5}{6}e^4 \]

and

\[ P(8) = 5e^4 \]

Thus, the relative rate of change of \( P \) at \( t = 8 \) is \( \frac{1}{6} \) which corresponds to a percentage rate of change of \( 16\frac{2}{3}\% \).
Recall that the demand function $f(p)$ for some commodity gives the maximal number of objects that can be sold at price $p$.

**Definition**

The *elasticity of demand* at price $p$, $E(p)$, is the additive inverse of the ratio of the relative rate of change of the demand by the relative rate of change of the price.
Elasticity of demand in terms of demand

From the definition of elasticity of demand, we have

\[ E(p) = -\left( \frac{f'(p)}{f(p)} \right) / \left( \frac{d}{dp} (p) \right) \]

\[ = -\frac{f'(p)}{f(p)} / \frac{1}{p} \]

\[ = -\frac{pf'(p)}{f(p)} \]
Example

Suppose that the demand function is given by $f(p) = pe^{-1.2p}$. Compute $E(5)$. What does it mean?
We compute using the product rule that

\[ f'(p) = e^{-1.2p} - 1.2pe^{-1.2p} = (1 - 1.2p)e^{-1.2p} \]

So, the elasticity of demand is given by

\[
E(p) = \frac{-pf'(p)}{f(p)} = \frac{p(1.2p - 1)e^{-1.2p}}{pe^{-1.2p}} = 1.2p - 1
\]

So, \(E(5) = 5\). That is, at a price of \(p = 5\), the rate at which the demand decreases is five times the rate of the price rise.
Marginal revenue in terms of elasticity of demand

Given a demand function of $f(p)$, the revenue produced by selling at the maximal allowed price is

$$R(p) = p \cdot f(p)$$

**Question**

What is the marginal revenue?
Solution

\[ R'(p) = \frac{d}{dp}(pf(p)) \]

\[ = f(p) + pf'(p) \]

\[ = f(p)[1 + \frac{pf'(p)}{f(p)}] \]

\[ = f(p)[1 - E(p)] \]

So, the marginal revenue is positive when \( E(p) < 1 \) and is negative when \( E(p) > 1 \). That is, one can increase revenues by raising prices when the elasticity of demand is less than one and one decreases revenues by raising prices when the elasticity of demand is greater than one.
Elastic and inelastic prices

**Definition**
A price $p$ is **elastic** if $E(p) > 1$ and is **inelastic** if $E(p) < 1$.

From our expression $R'(p) = f(p)[1 - E(p)]$ for the marginal revenue, we see that when the price is elastic, an increase in the price leads to a reduction of the revenue whilst for inelastic prices an increase in the price produces greater revenue.
The demand function for a certain commodity is given by $f(p) = 400\sqrt{95 - p}$ and the current price is $70. Is this price elastic? How should the price be changed to increase revenue?
Writing $E(p)$ for the elasticity of demand,

$$E(p) = \frac{-pf'(p)}{f(p)}$$

$$= -p \frac{d}{dp} \ln f(p)$$

$$= -p \frac{d}{dp} \ln(400(95 - p)^{\frac{1}{2}})$$

$$= -p \frac{d}{dp}(\ln(400) + \frac{1}{2} \ln(95 - p))$$

$$= -p \frac{-1}{190 - 2p}$$

Hence, $E(p) = \frac{p}{190-2p}$ so that $E(70) = \frac{70}{50} = 1.4 > 1$. That is, the current price is elastic and should be lowered to increase revenue.