1 (page 161, #26) Suppose that every member of $A$ has cardinality at most $\kappa$, then $\text{card} \bigcup A \leq (\text{card} A) \cdot \kappa$.

**Proof.** Define $Y := \{ (a, x) : x \in a \in A \}$. Then by definition of the union, the function $f : Y \rightarrow \bigcup A$ defined by $(a, x) \mapsto x$ is surjective. Hence, $A \simeq Y$. It suffices to show that $Y \leq A \times \kappa$. Let $R := \{ (a, f) \in A \times (\bigcup A \times \kappa) : f$ is an injective function with domain $a$ and range contained in $\kappa \}$. By hypothesis, every $a \in A$ has cardinality at most $\kappa$ so that there is some injective $f : a \rightarrow \kappa$. That is, $\text{dom} R = A$. By the axiom of choice there is a function $F \subseteq R$ with $\text{dom} F = \text{dom} R = A$.

Define $G : Y \rightarrow A \times \kappa$ by $(a, x) \mapsto (a, F(a)(x))$. If $y = (a, x)$ and $y' = (a', x')$ are two elements of $Y$ with $G(y) = G(y')$, then $a = a'$ and $F(a)(x) = F(a')(x') = F(a)(x')$. As $F(a)$ is injective, we have $x = x'$ as well, so that $y = y'$. Thus, $G$ is injective showing that $Y \leq A \times \kappa$.

Therefore, $\text{card} \bigcup A \leq (\text{card} A) \cdot \kappa$. \hfill \Box

2 (page 161, #27)

(a) Let $A$ be a collection of circular disks in the plane, no two of which intersect. Show that $A$ is countable.

**Proof.** As $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, if $D$ is any disk in the plane, then we must have $D \cap \mathbb{Q}^2 \neq \emptyset$. Let $R := \bigcup \{ D \cap \mathbb{Q}^2 | D \in A \}$. For $x \in R$, by definition, there is some $D \in A$ with $x \in D$. As the disks in $A$ are disjoint, there can be at most one such $D$. Define $G : R \rightarrow A$ by $G := \{ (x, D) \in \mathbb{Q}^2 \times A | x \in D \}$. The above remark shows that $R$ is a surjective function. Hence, we may find some injective $f : A \rightarrow R$ with $G \circ f = \text{id}_A$. As $R \subseteq \mathbb{Q}^2$, we have $|R| \leq \aleph_0$. Hence, $|A| \leq |\aleph_0|$. \hfill \Box

(b) Let $B$ be a collection of circles in the plane, no two of which intersect. Need $B$ be countable?

No. For $r$ a positive real number, let $S_r := \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = r \}$. Then $S_r$ is a circle and for $r \neq s$ we have $S_r \cap S_s = \emptyset$. Set $C := \{ S_r | r \in \mathbb{R}, r > 0 \}$.

(c) Let $C$ be a collection of figure eights in the plane, no two of which intersect. Need $C$ be countable?

Yes. Recall that we defined a figure eight to be a set of the form $X \cup Y$ where $X$ and $Y$ are circles whose bounded disks meet at exactly one point. For each pair of positive rational numbers $p$ and $q$ with $p \leq q$, we define $C_{p, q}$ to be the set of figure eights in $C$ whose smaller circle has radius $r$ with $p \leq r$ and whose large circle has radius $R$ with $q \leq R < q + p$. Note that $C = \bigcup_{p \leq q} C_{p, q}$ is a countable union of the sets $C_{p, q}$. If we show that each $C_{p, q}$ is countable, then we know that $C$ itself is. Note that if $F, E \in C_{p, q}$ are two distinct figure eights with the specified conditions on their radii, then not only are $F$ and $E$ disjoint, but the disks that they bound are also
disjoint as if \( F \) were a subset of the union of the disks bounded by \( F \), we would need the sum of the radii of the circles of \( F \) to be less than the radius of the larger circle of \( E \), but this sum is at least \( p + q \). Thus, exactly as in part (a) we may define an injective function \( f : C_{p,q} \to \mathbb{Q}^2 \) showing that \( C_{p,q} \) is countable.

3 (page 161, # 28) Find a set \( A \) of open intervals in \( \mathbb{R} \) such that every rational number belongs to one of those intervals but \( \bigcup A \neq \mathbb{R} \).

Let \( A = \{ (\sqrt{2} + n, \sqrt{2} + n + 1) : n \in \mathbb{Z} \} \).

4 (page 161, # 29) Let \( A \) be a set of positive real numbers. Assume that there is a bound \( b \) such that the sum of any finite subset of \( A \) is less than \( b \). Show that \( A \) is countable.

Proof. For each positive integer \( n \) define \( A_n := A \cap (\frac{1}{n}, \infty) \). I claim that for each \( n \), the set \( A_n \) is finite. Suppose this fails for some \( n \). Let \( k \in \omega \) be a natural number for which \( k > nb \). If \( A_n \) were infinite, then we could find \( a_1, \ldots, a_k \in A \) distinct.

We have \( b > \sum_{i=1}^{k} a_i > \sum_{i=1}^{k} \frac{1}{n} = \frac{k}{n} > \frac{nk}{n} = b \). With this contradiction, we see that \( A_n \) is finite. Thus, \( A = \bigcup_{n=1}^{\infty} A_n \) being a countable union of at most countable sets is itself at most countable. \( \square \)

5 (page 161, # 30) Assume that \( A \) is a set with at least two elements. Show that \( \text{Sq}(A) \preceq \omega A \).

Proof. By definition of \( \text{Sq}(A) \), we have \( \text{Sq}(A) = \bigcup_{n \in \omega} n A \).

We break the argument into two cases at this point.

If \( A \) is finite, then each of the sets \( n A \) is also finite and, thus, at most countable. Therefore, \( \text{Sq}(A) \) is also at most countable. However, as \( 2 \preceq A \), we see that \( \text{Sq}(A) \preceq \omega \times 2 \preceq \omega A \).

If \( A \) is infinite, then \( A \approx A \cup \{ \ast \} := A' \) where \( \ast \notin A \). Let \( g : A' \to A \) be a bijection and let \( G : \omega A' \to A' \) be the induced bijection given by \( \sigma \mapsto g \circ \sigma \).

Define a function \( F : \text{Sq}(A) \to \omega A' \) by

\[
F(\sigma)(n) = \begin{cases} 
\sigma(n) & \text{if } n \in \text{dom} \sigma \\
\ast & \text{otherwise}
\end{cases}
\]

Then \( F \) is injective as if \( F(\sigma) = F(\tau) \) we see that \( \text{dom}(\sigma) = F(\sigma)^{-1}[\text{dom}(\tau)] = F(\tau)^{-1}[\text{dom}(\sigma)] \) and that we have \( \sigma = F(\sigma) \upharpoonright \text{dom}(\sigma) = F(\tau) \upharpoonright \text{dom}(\sigma) = F(\tau) \upharpoonright \text{dom}(\tau) = \tau \).

Composing \( F \) with \( G \), we see that \( \text{Sq}(A) \preceq \omega A \). \( \square \)

6 (page 165, # 32) Let \( \mathcal{F}A \) be the collection of all finite subsets of \( A \). Show that if \( A \) is infinite, then \( A \approx \mathcal{F}A \).

Proof. We claim that for \( n \in \omega \) and \( n > 0 \) we have \( A \approx n A \). We prove this by induction on \( n \). For \( n = 1 \), there is an obvious bijection between \( A \) and \( 1 A \) given
by $a \mapsto [0 \mapsto a]$. Assuming the result for $n$, we compute

\[
\begin{align*}
{n^+ A} &= n \cup \{n\} A \\
&\approx n A \times \langle n \rangle A \quad \text{by Thm 6I(4)} \\
&\approx n A \times A \quad \text{by the base case} \\
&\approx A \times A \quad \text{by IH} \\
&\approx A \quad \text{by Lemma 6R}
\end{align*}
\]

Thus, by problem 26 of page 161, we have that $|\operatorname{Sq}(A)| \leq \aleph_0 \cdot |A| = |A|$. However, as $A \subseteq \operatorname{Sq}(A)$, we certainly have $|A| \leq |\operatorname{Sq}(A)|$. Therefore, $A \approx \operatorname{Sq}(A)$.

Now, define a function $\operatorname{Sq}(A) \to FA$ by $\sigma \mapsto \operatorname{rng} \sigma$. By definition of a finite set, this function is surjective. Thus, $|FA| \leq |\operatorname{Sq}(A)| = |A|$. Again, as every singleton in $A$ is a finite set, we have $|A| \leq |FA|$. Therefore, $A \approx FA$. $\square$

7 (page 165, # 35) Find a collection $A$ of $2^{\aleph_0}$ sets of natural numbers such that any two distinct members of $A$ have finite intersection.

**Proof.** We prove a claim.

**Claim:** Let $K$ be any infinite set with $\text{card}K = \kappa$. Let $IK := \{X \in \mathcal{P}K \mid |X| \geq \aleph_0\}$. Then $|IK| = 2^\kappa$.

**Proof of claim:** We may write $\mathcal{P}K$ as the disjoint union of $IK$ and $FK$. By the result of the previous problem, we have $\text{card}FA = \kappa$. Letting $\lambda := \text{card}IK$, we compute

\[
2^\kappa = \text{card}\mathcal{P}K = \text{card}(FK \cup IK) = \text{card}(FK) + \text{card}(IK) = \kappa + \lambda \quad \text{by Problem 6}
\]

\[
= \max\{\kappa, \lambda\} \quad \text{by the absorption law}
\]

\[
= \lambda \quad \text{as } \kappa < 2^\kappa
\]

By Euclid’s theorem, the set $P$ of prime numbers is a countably infinite set. Hence, the set $IP$ of infinite sets of primes has cardinality $2^{\aleph_0}$.

For each $X \in IP$ define $A_X := \{\prod_{p \leq n, p \in X} p \mid n \in \omega\} \subseteq \omega$. This association defined a function $F : IP \to \mathcal{P}\omega$. Let $A := \text{rng}F$.

Note that for any $X \in IP$ the set $A_X$ is infinite as $X$ is infinite so that there are elements of $A_X$ with arbitrarily many prime divisors. If $X \neq Y$ are two distinct elements of $IP$, then there is some prime which belongs to one set but not the other. Without loss of generality, we may assume $p \in X \setminus Y$. Let $A'_X := \{\prod_{p \leq n, p \in X} \ell \mid n < p\}$ and $A''_X := A_X \setminus A'_X$. Note that if $y \in A_Y$, then $p$ does not divide $y$ while $p$ divides every element of $A''_X$. Thus, $A_X \cap A_Y$ is contained in $A'_X \cap A_Y \subseteq A'_X \subseteq \{m \in \omega : m < p!\}$ which is a finite set. Hence, $A_X \cap A_Y$ is finite. As $A_X$ is infinite, this implies in particular that $A_X \neq A_Y$. Thus, the map $F : IP \to A$ is a bijection so that $A$ has cardinality $2^{\aleph_0}$ and is almost disjoint. $\square$

8 Show that for any infinite cardinal $\kappa$, we have $\kappa! = 2^\kappa$. 

Proof. Let $K$ be any set with $\text{card}K = \kappa$. Let $\text{Sym}(K) := \{\sigma : K \to K| \sigma \text{ a bijection }\}$. Then by definition, we have $\kappa! = \text{card} \text{Sym}(K)$.

As every permutation is a function from $K$ to $K$ (and in particular a relation with field $K$), we have $\text{Sym}(K) \subseteq \mathcal{P}(K \times K)$ so that $\kappa! \leq 2^{\kappa \cdot \kappa} = 2^\kappa$.

For the other inequality, define a function $F : \text{Sym}(K) \to \mathcal{P}K$ by $\sigma \mapsto \{x \in K| \sigma(x) = x\}$. Let $A := \text{rng}F$. To compute the cardinality of $A$ we need a claim.

Claim: Let $X$ be any set with $\text{card}X \geq 2$. Then there is a permutation $\pi : X \to X$ having no fixed points.

Proof of claim: If $X$ is finite, then $X \approx n$ for some $n \in \omega$ with $n > 2$. Let $f : n \to X$ be a bijection. Define $\sigma : n \to n$ by $\sigma(i) = i^+$ if $i < n - 1$ and $\sigma(n - 1) = 0$. Define $\pi : X \to X$ by $\pi = f \circ \pi \circ f^{-1}$.

If $X$ is infinite of cardinality $\lambda$, then as $\lambda = \lambda \times \lambda$, we can find a bijection $g : X \to X \times 2$. Define $\sigma : X \times 2 \to X \times 2$ by $(x, i) \mapsto (x, 1 - i)$. Let $\pi := g^{-1} \circ \sigma \circ g$.

From the claim it follows that $B = \{X \in \mathcal{P}K | \text{card}(K \setminus X) \neq 1\}$. For if $X = K$, then $X = F(\text{id}_K)$ and if $K \setminus X =: Y$ has more than one element, then we can find (by the claim) a permutation $\pi : Y \to Y$ having no fixed points so that if $\sigma = \pi \cup \text{id}_X$ we have $F(\sigma) = X$. Finally, if $K \setminus X = \{a\}$ is a singleton, then $X$ cannot be in the image of $F$ as any permutation which fixes $X$ must fix its complement setwise, and in the case of a singleton, being fixed setwise is the same as being fixed pointwise.

There is an obvious bijection between $K$ and the set of subsets of $K$ whose complements are singletons given by $a \mapsto K \setminus \{a\}$. Hence, $|\mathcal{P}K \setminus B| = \kappa$. As in the proof the claim in problem 7, we conclude that $|B| = 2^\kappa$. As $F : \text{Sym}(K) \to B$ is surjective, we conclude that $\kappa! \geq 2^\kappa$.

Combining this with the other inequality, we conclude that $\kappa! = 2^\kappa$. □