1. (page 83, # 13) Suppose \( m, n \in \omega \) and \( m \cdot n = 0 \). We show that \( m = 0 \) or \( n = 0 \).

**Proof.** Suppose that \( m \neq 0 \) and that \( n \neq 0 \). As we have already shown, there are \( k, \ell \in \omega \) with \( k^+ = m \) and \( \ell^+ = n \). Thus,

\[
m \cdot n = m \cdot (\ell^+)
\]
\[
= m \cdot \ell + m
\]
\[
= m \cdot \ell + k^+
\]
\[
= (m \cdot \ell + k)^+
\]
\[
\neq 0
\]

where the last inequality holds as 0 is not a successor. □

2. (page 83, # 17) We show that for natural numbers \( m, n, \) and \( p \in \omega \) the equality \( m^{n+p} = m^n \cdot m^p \) holds.

**Proof.** Fix \( m \) and \( n \in \omega \). Let \( I := \{ p \in \omega \mid m^{n+p} = m^n \cdot m^p \} \). We show that \( I \) is inductive. We compute

\[
m^{n+0} = m^n
\]
\[
= 0 + m^n
\]
\[
= m^n \cdot (0^+)
\]
\[
= m^n \cdot m^0
\]

Thus, \( 0 \in I \). Assume now that \( p \in I \). We compute

\[
m^{n+p^+} = m^{(n+p)^+}
\]
\[
= m^{n+p} \cdot m
\]
\[
= (m^n \cdot m^p) \cdot m
\]
\[
= m^n \cdot (m^p \cdot m)
\]
\[
= m^n \cdot m^{p^+}
\]

Thus, \( p \in I \Rightarrow p^+ \in I \). Therefore, \( I \) is inductive so that the asserted identity holds uniformly. □

3. (page 88, # 18)
\[
\epsilon_\omega^{-1}[\{7, 8\}] = 8
\]

4. (page 88, # 19) Let \( m \in \omega \) and \( d \in \omega \setminus \{0\} \). Then there are \( q, r \in \omega \) with \( m = (q \cdot d) + r \) and \( r < d \).

Before solving this problem, we prove a lemma.

**Lemma** Let \( m, n \in \omega \) be two natural numbers. If \( m < n \), then either \( m^+ = n \) or \( m^+ < n \).
Proof. By the trichotomy for $<$ on $\omega$, if the conclusion of this lemma were to fail, then $n < m^+$. By the definition of $<$, we would have $n \in m^+ = m \cup \{m\}$. By the definition of $\cup$, either $n \in m$ or $n = m$. In either case, we conclude by transitivity of $<$ that $m < m$ contradicting irreflexivity of $<$. \hfill \Box

We are in a position to prove the main proposition now.

Proof. Fix $d \in \omega \setminus \{0\}$. Let $I := \{m \in \omega \mid (\exists q, r \in \omega) m = (q \cdot d) + r \text{ and } r < d \}$. We show that $I$ is inductive. As $0 = 0 + 0 = (0 \cdot d) + 0$, we see that $0 \in I$. Suppose now that $m \in I$ witnessed by $m = (q \cdot d) + r$. By the lemma, we break into two cases: $r^+ < d$ or $r^+ = d$.

\textbf{Case 1:} $(r^+ < d)$ \quad $m^+ = ((q \cdot d) + r)^+ = (q \cdot d) + (r^+) \text{ so that } m^+ \in I$.

\textbf{Case 2:} $(r^+ = d)$ \quad $m^+ = ((q \cdot d) + r) + (r^+) = (q \cdot d) + d = q \cdot (d^+) = (q \cdot (d^+) + 0)$

In either case, we see that $m^+ \in I$ so that $I$ is inductive. \hfill \Box

5. (page 88, # 24) Assume $m, n, p, q \in \omega$ are natural numbers and that $m+n = p+q$. We show that $m \in p \iff q \in n$.

We prove a lemma from which the solution to this problem follows.

**Lemma** Let $a, b \in \omega$ Then $a < b$ if and only if there is some $c \in \omega$ with $a+(c^+) = b$.

Proof. We prove the $\Rightarrow$ direction first. Fix $a \in \omega$. Let $I := \{b \in \omega \mid b \leq a \text{ or } (\exists c \in \omega) a + (c^+) = b\}$. We show that $I$ is inductive. As $(\forall x \in \omega) 0 \leq x$, we know that $0 \in I$. Suppose that $b \in I$. If $b < a$, then $b^+ \leq a$ by the lemma of problem # 4. If $b = a$, then $b^+ = a^+ = a + 1 = a + (0^+) \text{ so that } b^+ \in I$. Finally, if $b > a$, then as $b \in I$ there is some $c \in \omega$ with $a + (c^+) = b$. So that $a + (c^+) = a + (c^+) = b$. In any case, $b^+ \in I$. Therefore, $I$ is inductive.

Now, we prove the converse implication. Fix $a \in \omega$. Let $J := \{b \in \omega \mid a < a+b^+\}$. We show that $J$ is inductive. We compute $a + 0^+ = (a + 0)^+ = a^+ = a \cup \{a\}$ so that we have $a \in (a + 0^+)$ meaning that $0 \in J$. Suppose now that $b \in J$. Then $a + (b^+) + (a + b^+) > (a + b^+) > a$. Thus, $b^+ \in J$. So, $J$ is inductive. \hfill \Box

We solve the main problem now.

Proof. Using commutativity of $+$, we see that it suffices to show that $m \in p \Rightarrow q \in n$. Suppose now that $m \in p$ and that $q \notin n$. Using the lemma, we write $p = m^+ + a$ and $q = n + b$ for some $a, b \in \omega$. We then have $m + n = (m + n) + (a^+ + b) = (m + n) + (a + b)^+ \text{ implying that } m + n < m + n$ which is a contradiction. \hfill \Box

6. (page 88, # 27) Assume that $A$ is a set; $G$ is a function; $f_1 : \omega \rightarrow A; f_2 : \omega \rightarrow A$; for each $n \in \omega$ we have $f \upharpoonright n \in \text{dom}G$, $f_2 \upharpoonright n \in \text{dom}G$, $f_1(n) = G(f_1 \upharpoonright n)$, and $f_2(n) = G(f_2 \upharpoonright n)$.

Then, $f_1 = f_2$.

Proof. Let $I := \{n \in \omega \mid f_1 \upharpoonright n = f_2 \upharpoonright n\}$. As $\text{dom}f_1 = \omega = \text{dom}f_2$, if $f_1 \neq f_2$, then there is some $n \in \omega$ with $f_1(n) \neq f_2(n)$ so that $f_1 \upharpoonright (n^+) \neq f_2 \upharpoonright (n^+)$ (meaning that $n^+ \notin I$). So, it suffices to show that $I$ is inductive.

As $0 = \emptyset$, we have $f_1 \upharpoonright 0 = \emptyset = f_2 \upharpoonright 0$. Thus, $0 \in I$. 

\[ \Box \]
Suppose now that \( n \in I \). We compute,

\[
\begin{align*}
f_1 \upharpoonright n^+ & = f_1 \upharpoonright (n \cup \{n\}) \\
& = (f_1 \upharpoonright n) \cup (f_1 \upharpoonright \{n\}) \\
& = (f_1 \upharpoonright n) \cup \{(n, f_1(n))\} \\
& = (f_1 \upharpoonright n) \cup \{(n, G(f_1 \upharpoonright n))\} \\
& = (f_2 \upharpoonright n) \cup \{(n, G(f_2 \upharpoonright n))\} \\
& = (f_2 \upharpoonright n) \cup \{(n, f_2(n))\} \\
& = (f_2 \upharpoonright n) \cup (f_2 \upharpoonright \{n\}) \\
& = f_2 \upharpoonright n^+
\end{align*}
\]

Thus, \( n^+ \in I \) and \( I \) is inductive. \( \square \)

7. (page 88, # 32)
(a) \( A^+ = \{1, \{1\}\} \) and \( \bigcup (A^+) = 2 \)
(b) \( \bigcup (\{2\}^+) = 3 \)

8. (page 89, # 34)
(a) \( \{0, 1, \{1\}\} \) is transitive.
(b) \( \{1\} \) is not transitive as \( 0 \in 1 \in \{1\} \) but \( 0 \notin \{1\} \).
(c) \( \langle 0, 1 \rangle \) is not transitive as \( 1 \in \{1\} \in \{0, \{1\}\} = \{0, 1\} \), but \( 1 \notin \langle 0, 1 \rangle \).