1. (page 54, # 22)
(a) $A \subseteq B \Rightarrow F[A] \subseteq F[B]$

Proof. Let $x \in F[A]$. By definition, there is some $a \in A$ with $F(a) = x$. As $A \subseteq B$, we have $a \in B$ so that $x = F(a) \in F[B]$. As $x$ was arbitrary, we have shown that $F[A] \subseteq F[B]$. \qed

(b) $(F \circ G)[A] = F[G[A]]$

Proof. We show left to right inclusion first. Take $x \in (F \circ G)[A]$. By definition, there is some $a \in A$ with $(F \circ G)(a) = x$. By definition of the composition, we have $x = (F \circ G)(a) = F(G(a))$. By definition of $G[A]$ we have that $G(a) \in G[A]$ so that $x \in F[G[A]]$ as claimed.

For the other inclusion, suppose that $x \in F[G[A]]$. By definition of $F[G[A]]$ there is some $y \in G[A]$ with $F(y) = x$. By definition of $G[A]$ there is some $a \in A$ with $G(a) = y$. Thus, $x = F(G(a)) = (F \circ G)(a)$ so that $x \in (F \circ G)[A]$.

With the two inclusions proved, we conclude that equality holds. \qed

(c) $Q \upharpoonright (A \cup B) = (Q \upharpoonright A) \cup (Q \upharpoonright B)$

Proof. Recall that for a general relation $R$ and set $A$ we define $R \upharpoonright A := (R \cap (A \times \text{rng}R))$.

$$Q \upharpoonright (A \cup B) = Q \cap ((A \cup B) \times \text{rng}Q)$$
$$= Q \cap ((A \times \text{rng}Q) \cup (B \times \text{rng}Q))$$
$$= (Q \cap (A \times \text{rng}Q)) \cup (Q \cap (B \times \text{rng}Q))$$
$$= (Q \upharpoonright A) \cup (Q \upharpoonright B)$$ \qed

2. (page 54, # 29) Assume $f : A \to B$ and $G : B \to \mathcal{P}A$ is defined by

$$G(b) := \{x \in A | f(x) = b\}$$

If $f : A \to B$ is surjective, then $G$ is injective.

Proof. Suppose that $b, b' \in B$ and $G(b) = G(b')$. As $f$ is surjective, there is some $a \in A$ with $f(a) = b$. By definition of $G$, we have $a \in G(b)$. As $G(b) = G(b')$, we have $a \in G(b')$ as well. By definition of $G$, we have $f(a) = b'$. Combining this with the equality $b = f(a)$, we conclude that $b = b'$. Thus, $G$ is injective. \qed

The converse does not hold. The correct biconditional is that $G$ is injective if and only if there is at most one element in $B \setminus f[A]$.

3. (page 54, # 30)
We need a lemma for this problem.
Lemma For any set \( X \) and nonempty set \( Y \), we have \( X \subseteq \bigcap Y \iff (\forall Z \in Y) X \subseteq Z \).

Proof. Suppose that \( X \subseteq \bigcap Y \). Let \( x \in X \). As \( X \subseteq \bigcap Y \), we have \( x \in \bigcap Y \). By definition of \( \bigcap Y \), for every \( Z \in Y \) we have \( x \in Z \). Thus, for every \( Z \in Y \) we have \( X \subseteq Z \).

Suppose now that for every element \( Z \in Y \) we have \( X \subseteq Z \). Take \( x \in X \). Then, for every \( Z \in Y \) we have \( x \in Z \). By the definition of \( \bigcap Y \), we have \( x \in \bigcap Y \). Therefore, \( X \subseteq \bigcap Y \). \( \square \)

(a1) \( F(B) = B \)

Proof. We define \( \mathcal{X} := \{ X \subseteq A | F(X) \subseteq X \} \). In our notation \( B = \bigcap \mathcal{X} \).

Note that \( \mathcal{X} \neq \emptyset \) as \( A \in \mathcal{X} \). Let \( X \in \mathcal{X} \) be arbitrary. Then \( B \subseteq X \) by the Lemma. By monotonicity of \( F \), \( F(B) \subseteq F(X) \). By definition of \( \mathcal{X} \), we have \( F(X) \subseteq X \). By transitivity of \( \subseteq \), we have \( F(B) \subseteq X \). By the Lemma again, we have \( F(B) \subseteq \bigcap \mathcal{X} = B \). By monotonicity of \( F \), we conclude that \( F(F(B)) \subseteq F(B) \). By definition of \( \mathcal{X} \), we have \( F(B) \in \mathcal{X} \). Applying the Lemma again, we have \( B \subseteq F(B) \). With both inclusions proved, we have \( F(B) = B \). \( \square \)

(a2) \( F(C) = C \)

Proof. Define a new function \( G : \mathcal{P}A \to \mathcal{P}A \) by \( G(X) := F(X) \). (Here \( \mathcal{X} := A \setminus \{ X \} \).

We check that \( G \) is monotone: if \( X \subseteq Y \), then \( Y \subseteq X \). As \( F \) is monotone, we have \( F(Y) \subseteq F(X) \). Taking complements again, we have \( G(X) = \overline{F(X)} \subseteq \overline{F(Y)} = G(Y) \).

Set \( D := \bigcap \{ X \subseteq A | G(X) \subseteq X \} \). By part (a1), we know \( G(D) = D \).

Using De Morgan’s laws and the definitions of \( G \) and \( D \), we compute:

\[
\overline{C} = \bigcup \{ X \subseteq A | F(X) \supseteq X \} \\
= \bigcap \{ \overline{X} | F(X) \supseteq X \} \\
= \bigcap \{ X | F(X) \supseteq \overline{X} \} \\
= \bigcap \{ X | F(X) \subseteq X \} \\
= D
\]

Use the definition of \( G \) to see that

\[
\overline{C} = D \\
= G(D) \\
= \overline{F(D)} \\
= \overline{F( \overline{C} )} \\
= \overline{F(C)}
\]

Taking complements again, we conclude that \( C = F(C) \). \( \square \)

(b) Suppose that \( F(X) = X \). We are to show that \( B \subseteq X \subseteq C \).
Proof. If \( F(X) = X \), then in particular, \( F(X) \subseteq X \) so that \( X \in X \). By the Lemma, \( B \subseteq X \).

Likewise, if \( X \subseteq F(X) \), then as \( C = \bigcup \{ Y \subseteq A | Y \subseteq F(Y) \} \) by the dual form of the Lemma, \( X \subseteq C \). \( \square \)

4. (page 55, # 31) Suppose that the first form of the axiom of choice holds. Let \( I \) be any set and \( H \) a function whose domain includes \( I \). Recall that \( \prod_{i \in I} H(i) := \{ f \mid f \text{ is a function with } \text{dom}(f) = I \text{ & } (\forall i \in I) f(i) \in H(i) \} \). Let \( Y := \bigcup_{i \in I} H(i) = \text{rng}(H \upharpoonright I) \). Let \( R := \{ (i, y) \in I \times Y | y \in H(i) \} \).

By definition of the domain, \( H(i) \neq \emptyset \) for each \( i \in I \). By the Axiom of Choice in its first form, there is some function \( f \subseteq R \) with \( \text{dom}f = \text{dom}R = I \).

Comparing the definitions of \( R \) and \( \prod_{i \in I} H(i) \) we see that \( f \in \prod_{i \in I} H(i) \) showing that \( \prod_{i \in I} H(i) \neq \emptyset \).

Suppose now that the second form of the Axiom of Choice holds. That is, every Cartesian product of nonempty sets is nonempty. Let \( R \) be any relation. If \( R = \emptyset \), then \( R \) is already a function so that we may take \( f = R \) giving a function which is a subset of \( R \) having the same domain. So, we may assume that \( R \neq \emptyset \). Set \( I := \text{dom}R \). Let \( B := \text{rng}R \). Define \( H : I \to B \) by \( i \mapsto \{ y \in \text{rng}(R(i, y)) \in R \} \).

By definition of the domain, \( H(i) \neq \emptyset \) for each \( i \in I \). By the Axiom of Choice in its second form, there is some \( f \in \prod_{i \in I} H(i) \).

By definition of the product, \( f : I \to Y \) and \( (\forall i \in I) f(i) \in H(i) \). By definition of \( H \), we have that for each \( i \in I \), \( (\langle i, f(i) \rangle \in R \).

That is, \( f \subseteq R \) and \( \text{dom}f = I \).

Thus, \( f \) witnesses that this instance of the Axiom of Choice in its first form holds.

5. (page 61, # 32)

(a) \( R \) is symmetric iff \( R^{-1} \subseteq R \)

Proof. Suppose \( R \) is symmetric. Let \( (x, y) \in R^{-1} \). By definition of the converse relation, \( (y, x) \in R \). As \( R \) is symmetric, we conclude \( (x, y) \in R \). Thus, \( R^{-1} \subseteq R \).

Conversely, suppose that \( R^{-1} \subseteq R \). Suppose that \( (x, y) \in R \). By definition of \( R^{-1} \), we have \( (y, x) \in R^{-1} \). As \( R^{-1} \subseteq R \), we have \( (y, x) \in R \). Therefore, \( R \) is symmetric. \( \square \)

(b) \( R \) is transitive iff \( R \circ R \subseteq R \)

Proof. Suppose that \( R \) is transitive. Let \( t \in R \circ R \). By definition, there are \( x, y, \) and \( z \) such that \( t = (x, z) \), \( (x, y) \in R \) and \( (y, x) \in R \). As \( R \) is transitive, we know \( (x, z) \in R \). Therefore, \( t \in R \) so that \( R \circ R \subseteq R \).

Conversely, suppose that \( R \circ R \subseteq R \). Suppose that \( (x, y) \in R \) and that \( (y, z) \in R \). By definition of the composition, \( (x, z) \in R \circ R \). As \( R \circ R \) is a subset of \( R \), we have \( (x, z) \in R \). Therefore, \( R \) is transitive. \( \square \)

6. (page 61, # 37) We prove that \( R_{II} \) is an equivalence relation on \( A \). First, we check that \( R_{II} \) is reflexive. Take \( a \in A \). Then as \( A \) is a partition, there is some \( B \in A \) with \( a \in B \). From the definition of \( R_{II} \) (applied to \( x = y = a \) and taking \( B \) as the witness) we have \( a R_{II} a \). Secondly, we check that \( R_{II} \) is symmetric. Suppose that \( aR_{II} b \). So there is some \( B \in A \) with \( a \in B \& b \in B \). As the conjunction is symmetric, we have \( b \in B \& a \in B \) so that \( b R_{II} a \) as well showing that \( R_{II} \) is symmetric. Finally, we check that \( R_{II} \) is transitive. Suppose that \( aR_{II} b \) and \( bR_{II} c \).
The first relation is witnessed by some $B \in \Pi$ with $a \in B$ and $b \in B$ while the second relation is witnessed by some $C \in \Pi$ with $b \in C$ and $c \in C$. As $\Pi$ is a partition, either $B = C$ or $B \cap C = \emptyset$. As $b \in B \cap C$, we must be in the first case: $B = C$. But then, $a \in B$ and $c \in B$ so that $aR_{\Pi}c$ holds.

7 (page 62, # 40) This question is not well-posed as one needs to define the notion of number of prime factors precisely. If one sets $P(x) := \{p|p \text{ is prime and divides } x\}$ and then defines the number of prime factors of $x$ to be the size of $P(x)$, then the answer to the question is no as, for instance, $2R3$ but $\neg(6R9)$. However, if one writes $n = \prod_{\text{prime } p} p^{v_p(n)}$ and then defines the number of prime factors of $n$ to be

\[ \sum_{\text{prime } p} v_p(n) \] the answer would be yes as $3n$ always has one more factor than $n$ so that $f$ would respect $R$.

8 (page 64, # 44) We prove now that $f$ is injective. Suppose that $f(x) = f(y)$. If $x \neq y$, then either $x < y$ or $y < x$. As the roles of $x$ and $y$ are interchangeable, we may assume that $x < y$. By hypothesis on $f$, $f(x) < f(y)$. As $f(y) = f(x)$ we would have $f(x) < f(x)$ contradicting Theorem 3R.

We prove now the second clause. Suppose that $f(x) < f(y)$. As $<$ is a total order, either $x < y$, $x = y$, or $x > y$. We eliminate the latter two possibilities. If $x = y$, then, of course, $f(x) = f(y) > f(x)$ violating the trichotomy (Theorem 3R). If $x > y$, then by monotonicity of $f$, we have $f(x) > f(y)$. By transitivity of $>$, we would have $f(x) < f(x)$ which again violates Theorem 3R. Therefore, we must have $x < y$. 