1. (page 207 # 26)

Proof. We show by transfinite induction on \(\alpha \in \text{ON}\) that \(\text{rank}\alpha = \alpha\). Take \(\alpha\) an ordinal and suppose that \((\forall \beta \in \alpha) \text{rank}\beta = \beta\). We calculate

\[
\text{rank}\alpha = \bigcup\{(\text{rank}\beta)^+ \mid \beta \in \alpha\} \text{ by Theorem 7V(b)}
\]

\[
= \bigcup\{\beta^+ \mid \beta \in \alpha\} \text{ by the inductive hypothesis}
\]

\[
= \alpha \quad \square
\]

2. (page 207, # 30)

Proof.

\[
\text{rank}\{a, b\} = \bigcup\{(\text{rank}\, x)^+ \mid x \in \{a, b\}\}
\]

\[
= \bigcup\{(\text{rank}\, a)^+, (\text{rank}\, b)^+\}
\]

\[
= \max\{(\text{rank}\, a)^+, (\text{rank}\, b)^+\}
\]

Before proving the next equation, we note that for any two sets \(X\) and \(Y\), if \(X \subseteq Y\), then \(\text{rank}\, X \leq \text{rank}\, Y\) as for any ordinal \(\alpha\), if \(Y \subseteq V_\alpha\), then \(X \subseteq Y \subseteq V_\alpha\).

\[
\text{rank}\, P_a = \bigcup\{(\text{rank}\, b)^+ \mid b \in P_a\}
\]

\[
= (\text{rank}\, a)^+ \text{ as } a \in P_a \text{ and } (\text{rank}\, b)^+ \leq (\text{rank}\, a)^+ \text{ for } b \in P_a
\]

Finally, if \(x \in \bigcup a\), then there is some \(y \in a\) such that \(x \in y\) so that \(\text{rank}\, x \in \text{rank}\, a\). Thus, for any \(x \in a\) we have \(\text{rank}\, x \in \text{rank}\, a\) so that for any \(x \in \bigcup a\) we have \((\text{rank}\, x)^+ \leq \text{rank}\, a\). Thus, \(\text{rank}\, \bigcup a = \bigcup\{(\text{rank}\, x)^+ \mid x \in \bigcup a\} \leq \text{rank}\, a\). \(\square\)

3. (page 208, # 35)

Proof. Suppose that \(a^+ = b^+, \text{ but } a \neq b\). We have \(a \in a^+ = b^+ = b \cup \{b\}\). Assuming, as we have that \(a \neq b\), we must have \(a \in b\). Likewise, \(b \in b^+ = a^+ = a \cup \{b\}\) so that \(b \in a\). But then \(\text{rank}\, a \in \text{rank}\, b \in \text{rank}\, a\) which is impossible for ordinals. \(\square\)

4. (page 208, # 36)

Proof. We have \(S \subseteq \text{TC}(S)\) so that \(\text{rank}\, S \leq \text{rank}\, \text{TC}(S)\).

We prove now the opposite inequality.

Using the notation of Problem 7 of page 178 (with \(C\) replaced by \(S\)) we show by induction on \(n\) that \(\text{rank}\, F(n) \leq \text{rank}\, S\). In the case of \(n = 0\) we have \(F(0) = S\) so that, of course, \(\text{rank}\, F(0) \leq \text{rank}\, S\). More generally, we have \(F(n^+) = F(n) \cup \bigcup F(n)\). By problem 2 and induction, we have \(\text{rank}\, \bigcup F(n) \leq \text{rank}\, F(n) \leq \text{rank}\, S\) so that \(\text{rank}\, F(n^+) = \max\{\text{rank}\, F(n), \text{rank}\, \bigcup F(n)\} \leq \text{rank}\, F(n) \leq \text{rank}\, S\).
Now, \( \text{TC}(S) = \bigcup_{n \in \omega} F(n) \) so that \( \text{rank TC}(S) = \text{rank} \bigcup_{n \in \omega} F(n) = \bigcup_{n \in \omega} \text{rank} F(n) \leq \text{rank} S \).

Thus, \( \text{rank TC}(S) = \text{rank} S \). \( \square \)

5. (page 219 # 3)

**Proof.** By monotonicity, we know that \( \beta \in \gamma \Rightarrow t_\beta \in t_\gamma \). Suppose now that \( \beta, \gamma \in \text{ON} \) and that \( t_\beta \in t_\gamma \). By the trichotomy for ordinals, we know \( \gamma \in \beta \), \( \gamma = \beta \), or \( \beta \in \gamma \).

In the first case, we have \( t_\gamma \in t_\beta \) but then by transitivity we would have \( t_\beta \in t_\beta \) which is impossible for ordinals. Likewise, in the second case we have the contradictory \( t_\beta \in t_\beta \). Thus, we must have \( \beta \in \gamma \).

As \( t \) assigns a unique ordinal \( t_\alpha \) to each ordinal \( \alpha \), we know \( \alpha = \beta \) implies that \( t_\alpha = t_\beta \). Suppose now that \( t_\alpha = t_\beta \). If \( \alpha \neq \beta \), then either \( \alpha \in \beta \) or \( \beta \in \alpha \). Without loss of generality, we may suppose \( \alpha \in \beta \). But then \( t_\alpha \in t_\beta = t_\alpha \) which is impossible. \( \square \)

6. (page 220 # 6)

**Proof.** Suppose that \( T := \{ t_\alpha \mid \alpha \in \text{ON} \} \) is bounded by some ordinal \( \beta \). Let \( \varphi(x, y) \) be the formula \( x = t_y \). By the above problem, for each \( x \) there is at most one \( y \) such that \( \varphi(x, y) \) holds. Thus, the class \( A = \{ \alpha \mid (\exists x \in \beta^+) \varphi(x, \alpha) \} \) is a set of ordinals. By hypothesis, every ordinal is an element of \( A \) but we know that no set contains every ordinal.

Let \( A \subseteq T \) be a subset of the class \( T \). Let \( S = \{ \alpha \in \text{ON} \mid (\exists x \in A) \varphi(x, \alpha) \} \). As above, \( S \) is a set. By Theorem Schema 8E, we have \( \text{sup} A = \text{sup} S \in T \). Thus, \( T \) is closed. \( \square \)

7. (page 220 # 7)

**Proof.** Let \( t \) and \( \beta \) be given as in Veblen’s fixed-point theorem. Let \( \gamma \) be the fixed point given by the proof in the text and let \( f: \omega \rightarrow \gamma \) be the function with \( f(0) = \beta \) and \( f(n^+) = t_{f(n)} \) for \( n \in \omega \). Recall that \( \gamma = \bigcup \{ f(n) : n \in \omega \} \). Suppose that \( \delta \geq \beta \) and \( t_\delta = \delta \). We show by induction on \( n \in \omega \) that \( f(n) \leq \delta \). By hypothesis, \( \delta \geq \beta = f(0) \). Assuming that \( f(n) \leq \delta \), then by monotonicity we have \( f(n^+) = t_{f(n)} \leq t_\delta = \delta \). Thus, \( (\forall n \in \omega) f(n) \leq \delta \), so that \( \gamma \), being the supremum of \( f(n) \) as \( n \) ranges through \( \omega \), is less than or equal to \( \delta \). \( \square \)

8. (page 220 # 8) Before proving that \( t' \) is a normal operation we should define it more carefully. We define \( t'_{\alpha} \) to be the least fixed point \( \gamma \) of \( t \) with \( \gamma \) strictly greater than \( t'_{\beta} \) for all \( \beta \in \alpha \).

**Proof.** Monotonicity is almost immediate for if \( \alpha \in \beta \), then we require \( t'_{\beta} \) to be strictly greater than \( t'_{\delta} \) for all \( \delta \in \beta \) which implies, in particular, that \( t'_{\delta} \ni t'_{\alpha} \).

For continuity, suppose that \( \lambda \) is a limit ordinal. Let \( \beta := \text{sup} \{ t'_{\alpha} \mid \alpha \in \lambda \} \). Then we have

\[
\begin{align*}
t_\beta &= \sup \{ t_\alpha \mid \alpha \in \lambda \} \text{ by Theorem Schema 8E} \\
&= \sup \{ t'_{\alpha} \mid \alpha \in \lambda \} \text{ as each } t'_{\alpha} \text{ is a fixed point of } t \\
&= \beta
\end{align*}
\]

Thus, \( \beta \) is a fixed point of \( t \). For each \( \alpha \in \lambda \) we have \( \alpha^+ \in \lambda \) as well so that \( t'_{\alpha} \in t'_{\alpha^+} \leq \beta \) implying that \( \beta \) is strictly greater than \( t'_{\alpha} \) for every \( \alpha \in \lambda \). Thus,
\( \beta \geq t'_{\lambda} \). But, by monotonicity, we cannot have \( t'_{\lambda} \) any smaller than \( \beta \). Thus, \( \beta = t'_{\lambda} \) so that \( t' \) is continuous. \( \square \)