MATH 125
AUTUMN 2005
SOLUTIONS TO THE SECOND MIDTERM

1. Consider the following sentences in the language with one constant symbol c, two binary function symbols, $\oplus$ and $*$, and one binary relation symbol R. We write $x \oplus y$ for $\oplus xy$, $x * y$ for $*xy$, and xRy for Rxy.

For each sentence exhibit a model of the sentence and a model of its negation.

a. $\forall u \forall v \exists x (\neg v \simeq c \Rightarrow u \oplus (v * x) \simeq c)$
   $(\mathbb{R}, +, \times, 0) \models a.$, $(\mathbb{Z}, \times, \times, 2) \models \neg a.$

b. $\forall x \forall y (xRy \Rightarrow \neg yRx)$
   $(\mathbb{Q}, <) \models b.$, $(\mathbb{Q}, =) \models \neg b.$

c. $\forall x \forall y \forall z (xRz \Rightarrow x * yRy * z)$
   $(\mathbb{R}, +, <) \models c.$, $(\mathbb{R}, \times, <) \models \neg c.$ (Consider the case of $y = -1$).

2. The *(finite) spectrum* of a theory T is is the set

$$\operatorname{Spec}(T) := \{n \in \mathbb{Z}_+ \mid \text{ there is a model } \mathfrak{M} \models T \text{ with } ||\mathfrak{M}|| = n\}$$

Let $\mathcal{L} := \mathcal{L}(E, F)$ be the language having two binary relation symbols E and F.

Find an $\mathcal{L}$-theory T with $\operatorname{Spec}(T) = \{n \in \mathbb{Z}_+ \mid n \text{ is composite}\}$.

Solution:

$$T := \{\forall x (Exx \wedge Fxx), \forall x \forall y (Exy \Rightarrow Eyx)\ , \forall x \forall y (Fxy \Rightarrow Fyx)\ ,$$
$$\forall x \forall y \forall z ((Exy \wedge Eyz) \Rightarrow Exz)\ , \forall x \forall y \forall z ((Fxy \wedge Fyz) \Rightarrow Fxz), \exists x \exists y (Exy \wedge \neg x \simeq y)\ ,$$
$$\exists x \exists y (Fxy \wedge \neg x \simeq y)\ , \forall x \forall y \exists z (Exz \wedge Fyz), \forall x \forall y ((Exy \wedge Fxy) \Rightarrow x \simeq y)\}$$

The first five axioms assert that E and F are equivalence relations. The next two axioms assert that there are at least two E-classes and at least two F-classes. The eighth axiom asserts that if $\mathfrak{M} \models T$, then the map $\pi : M \to (M/E) \times (M/F)$ given by $x \mapsto \langle [x]_E, [x]_F \rangle$ is surjective while the last axiom asserts that this map is injective. Hence, $|M| = |M/E| \cdot |M/F|$. As each of the numbers on the right is at least two, if they are finite, then $|M|$ is composite. Thus, $\operatorname{Spec}(T)$ consists only of composite numbers. Conversely, if n is composite, expressed as $n = m\ell$ with $2 \leq m, \ell < n$, then let M be the set $\{\langle i, j \rangle \in \mathbb{Z}^2 \mid 1 \leq i \leq m, 1 \leq j \leq \ell\}$ with $E^{\mathfrak{M}} := \{\langle \langle i, t \rangle, \langle i, s \rangle \rangle \mid 1 \leq i \leq m, 1 \leq s, t \leq \ell\}$ and $F^{\mathfrak{M}} := \{\langle \langle t, j \rangle, \langle s, j \rangle \rangle \mid 1 \leq s, t \leq m, 1 \leq j \leq \ell\}$. Clearly, $\mathfrak{M} \models T$ and $|M| = n$. Hence, every positive composite number belongs to the spectrum of T.

3. Fix a first-order language $\mathcal{L}$. Suppose that T is a theory and whenever $\mathfrak{M} \models T$ and $\mathfrak{N} \models T$, then $\mathfrak{M} \cong \mathfrak{N}$. Show that T generates a complete theory in the sense that for every $\mathcal{L}$-sentence ψ either $T \vdash^* \psi$ or $T \vdash^* \neg\psi$.

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Solution: Suppose that T does not generate a complete theory. Let ψ be an $\mathcal{L}$-sentence for which $T \not\vdash^* \psi$ and $T \not\vdash^* \neg\psi$. By definition of $\vdash^*$, as $T \not\vdash^* \psi$ there is a model $\mathfrak{M} \models T$ for which $\mathfrak{M} \not\models \psi$ and as $T \not\vdash^* \neg\psi$ there is a model $\mathfrak{N} \models T$ for which $\mathfrak{N} \not\models \neg\psi$. By the definition of the satisfaction of a negation, we have $\mathfrak{N} \models \neg\psi$. By hypothesis, $\mathfrak{M} \cong \mathfrak{N}$ as both structures are models of T. As these structures are isomorphic, they are elementarily equivalent, but $\mathfrak{M} \models \neg\psi$ while $\mathfrak{N} \not\models \neg\psi$. With this contradoction we conclude that T generates a complete theory.

4. Prove or disprove: If $\mathfrak{N} \succeq (\mathbb{N}, +, \times, 0, 1, <)$ is an elementary extension of the natural numbers considered in the language with constant symbols for zero and one, function symbols for addition and multiplication, and a binary relation symbol for the usual ordering on $\mathbb{N}$ and $X \subseteq N$ is a nonempty subset of the universe of $\mathfrak{N}$, then X has a ($<^{\mathfrak{N}}$-)least element.
Solution: The assertion is false. Indeed, if $\mathfrak{N}$ is any nonstandard model of the theory of true arithmetic, then the set

$$X := N \smallsetminus \mathbb{N} = \{a \in N \mid a \text{ is not of the form } 0^{\mathfrak{N}} \text{ or } \overbrace{1^{\mathfrak{N}} +^{\mathfrak{N}} \cdots +^{\mathfrak{N}} 1^{\mathfrak{N}}}^{n \text{ times}} \text{ for some } n \in \mathbb{Z}_+\}$$

is nonempty (this is the very definition of being nonstandard!) and has no least element: Let $a \in X$ be the purported least element. $\mathbb{N} \models \forall y(y \simeq 0 \vee \exists z(z + 1 \simeq y \wedge z < y))$. Hence, $\mathfrak{N} \models \forall y(y \simeq 0 \vee \exists z(z+1 \simeq y \wedge z < y))$. As $a \neq 0^{\mathfrak{N}}$, there must be some $b \in N$ with $b +^{\mathfrak{N}} 1^{\mathfrak{N}} = a$ and $b <^{\mathfrak{N}} a$. The element b cannot belong to X as a is supposed to be the least element of X, but if b is not an element of X then it is expressible as a natural number which would be the case for $a = b +^{\mathfrak{N}} 1^{\mathfrak{N}}$ as well, contrary to the definition of X. Hence, X has no least element.

5. Let the language $\mathcal{L}$ consist of a single unary function symbol f. Let $\varphi := (\forall x fffx \simeq x \wedge \forall y fy \not\simeq y)$.

Show that if $\mathfrak{M} \models \phi$ and $\mathfrak{N} \models \phi$ are two models of ϕ each of cardinality 27, then $\mathfrak{M} \cong \mathfrak{N}$.
Solution: Let $\mathfrak{M} = (M, \boldsymbol{f}) \models \phi$ be a model of ϕ. On M we define an equivalence relation $\sim$ by $x \sim y \leftrightarrow \boldsymbol{f}^n x = y$ for some positive integer n. We check that this is an equivalence relation. Let $x, y, z \in M$ be three elements of M. As $\boldsymbol{f}^3 x = x$, we have $x \sim x$. If $x \sim y$ witnessed by $\boldsymbol{f}^n x = y$, then writing $n = 3m + j$ with $m, j \in \mathbb{Z}$ we have $\boldsymbol{f}^{3-j} y = \boldsymbol{f}^{3-j} \boldsymbol{f}^{3m+j} x = \boldsymbol{f}^{3(m+1)} x = x$. Finally, if $x \sim y$ and $y \sim z$ witnessed by $\boldsymbol{f}^n x = y$ and $\boldsymbol{f}^m y = z$, we have $x \sim z$ witnessed by $\boldsymbol{f}^{m+n} x = z$.

The axioms imply that each $\sim$-class has exactly three elements. Indeed, if $x \in M$, $n \in \mathbb{Z}_+$ and $n = 3m + r$ with $0 \leq r < 3$, then $\boldsymbol{f}^n x = \boldsymbol{f}^r x$ so that the number of elements $\sim$-equivalent to x is at most three. By the second clause in the axiom, $x \neq \boldsymbol{f} x$, $\boldsymbol{f} x \neq \boldsymbol{f}^2 x$, and $x \neq \boldsymbol{f}^2 x$ as if $x = \boldsymbol{f}^2 x$, we would have $\boldsymbol{f} x = \boldsymbol{f}^3 x = x$.

Let now $\mathfrak{M} \models \phi$ and $\mathfrak{N} \models \phi$ be two models of ϕ of size twenty-seven. Pick $x_1, \ldots, x_9 \in M$ representatives of the $\sim$-equivalence classes in M and $y_1, \ldots, y_9$ representatives of the $\sim$-equivalence classes in N. By out above calculations, $M = \{(f^{\mathfrak{M}})^i(x_j) \mid i \in \{0, 1, 2\}, j \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}\}$ and $N = \{(f^{\mathfrak{N}})^i(y_j) \mid i \in \{0, 1, 2\}, j \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}\}$. Define $\alpha : \mathfrak{M} \to \mathfrak{N}$ by $\alpha((f^{\mathfrak{M}})^i(x_j)) := (f^{\mathfrak{N}})^i(y_j)$. As noted above, this map is well-defined and is a bijection. Moreover, it and its inverse respect the interpretation of f. Hence, $\mathfrak{M} \cong \mathfrak{N}$.