MATH 113: INTRODUCTION TO ABSTRACT ALGEBRA
AUTUMN 2007
FINAL EXAM PRACTICE PROBLEMS

1. Let $\alpha \in \mathbb{C}$ be a complex number satisfying the equation $\alpha^3 - 3\alpha + 1 = 0$. Compute $[\mathbb{Q}(\sqrt{-5}, \alpha) : \mathbb{Q}]$.

2. Prove or disprove: If K is an extension field of $\mathbb{Q}$ and $[K : \mathbb{Q}] < \infty$, then there is an irreducible polynomial $P(X) \in K[X]$.

3. Let $g(X) = X^4 - X^2 + X + 1 \in \mathbb{Z}_3[X]$. Write g as a product of irreducible polynomials.

4. Write $\frac{5-\sqrt[3]{49}}{1+\sqrt[3]{7}}$ in the form $a + b\sqrt[3]{7} + c\sqrt[3]{49}$ for rational numbers a, b, and c.

5. Show that if $\phi : R \to S$ is a homomorphism of commutative rings and $a \in S$ is any element, then there is a unique homomorphism $\widetilde{\phi} : R[X] \to S$ for which $\widetilde{\phi}(X) = a$ and $\widetilde{\phi}(r) = \phi(r)$ for all $r \in R$.

6. Express the quotient group $(\mathbb{Z}_{60} \times \mathbb{Z}_{24} \times \mathbb{Z}_{40})/\langle(5, 16, 25)\rangle$ as a direct sum of cyclic groups.

8. Compute $13^{5,389}$ in $\mathbb{Z}_{305}$.

9. Prove or disprove: If $\phi : R \to S$ is a homomorphism of rings and $\mathfrak{p} \subsetneq S$ is a prime ideal, then $\phi^{-1}\mathfrak{p} := \{x \in R : \phi(x) \in \mathfrak{p}\}$ is a prime ideal.

10. Find (with proof) all automorphisms of $\mathbb{Z}$.

11. Prove or disprove: every group of order twelve has a subgroup of order six.

12. Prove or disprove: If G is a nonempty set with a binary operations $*$ which satisfies left and right cancelation for all a and b in G there is some $x \in G$ with $a * x = b$ and some y with $y * a = b$, then $(G, *)$ is a group.

13. How many elements of the group $\mathbb{Z}_{12} \times \mathbb{Z}_{16} \times S_5$ have order four?

14. Prove or disprove: If $R = \mathcal{C}([0,1])$ is the ring of continuous real-valued functions on the closed interval $[0,1] := \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, then $I := \{f \in R : f(\frac{1}{2}) = 0\}$ is a maximal ideal.

15. How many subgroups of S_5 have exactly three elements?

16. Let G be the set of functions from $\mathbb{R}$ to $\mathbb{R}$ of the form $x \mapsto ax + b$ for some real numbers a and b with $a \neq 0$. Prove or disprove: G is a group under the binary operation of composition.

17. Write the following permutation as a product of disjoint cycles.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 8 & 2 & 6 & 9 & 7 & 1 & 3 & 4 \end{pmatrix}$$

18. Prove or disprove: If G is a group and $H < G$ is a subgroup with $\#(G/H) = 3$, then $H \lhd G$.

19. Prove or disprove: If K is a field and f and g are polynomials over K and $K[X]/(f) \cong K[X]/(g)$, then $(f) = (g)$.

20. Prove or disprove: There is a nontrivial homomorphism $\phi : \mathbb{Z}_4 \to S_3$

21. Let p be a prime number. Suppose that $\mathbb{Z}_p$ acts on the set X. Let $Y := \{x \in X : (\forall g \in \mathbb{Z}_p) g \cdot x = x\}$. Show that $\#Y \equiv \#X \pmod{p}$.

22. Let K be a field and $g(x) \in K[x] \smallsetminus K$ a nonconstant polynomial over K of degree d. Prove that there are at most d elements a of K satisfying $f(a) = 0$.

23. Let $g(x) := x^3 + x + 1 \in \mathbb{Z}_2$. Let $K := \mathbb{Z}_2[x]/(g)$. Prove that K is a field. Let $\alpha \in K$ be a solution to $\alpha^3 + \alpha + 1 = 0$. Write the polynomial $x^3 + \alpha + 1$ as a product of irreducible polynomials over K.

24. Prove or disprove: If $\phi : R \to S$ is a homomorphism of rings, $a \in R$ and $\phi(a) \in S^\times$, then $a \in R^\times$.

25. How many elements of the factor group $\mathbb{Q}/\mathbb{Z}$ have order *dividing* $5,239,290$?