MATH 113: INTRODUCTION TO ABSTRACT ALGEBRA
AUTUMN 2007
MIDTERM 2 PRACTICE PROBLEMS

1. Compute $3^{5,789,345}$ in $\mathbb{Z}_{70}$.

2. Prove or disprove: If G is a group and $K \trianglelefteq G$ and $N \trianglelefteq G$ are two normal subgroups which are isomorphic to each other, $N \cong K$, then $G/K \cong G/N$.

3. Let $R := \{f \mid f : \mathbb{Z} \to \mathbb{Z}\}$ be the set of functions from the integers to the integers. Define $+$ on R by $(f+g)(x) := f(x) + g(x)$ and $\cdot$ on R by $(f \cdot g)(x) = (f \circ g)(x) = f(g(x))$. Prove or disprove: $(R, +, \cdot)$ is a ring.

4. Prove or disprove: if G is a group of order 32, then there is a group H of order 16 and a homomorphism $\phi : G \to H$ which is onto.

5. Let $G = S_{\mathbb{R}}$ be the group of permutations of the real numbers. Let $H \leq G$ be the subgroup of G consisting of those permutations which fix all but finitely many points. That is, $\pi \in H \iff \{x \in \mathbb{R} \mid \pi(x) \neq x\}$ is finite. Is H a normal subgroup of G? Prove that your answer is correct.

6. Describe $(\mathbb{Z}_{12} \times \mathbb{Z}_3)/\langle(2,2)\rangle$.

7. Let R be an integral domain and $a, b, c \in R$ elements of R. Show that there are at most three elements x of R satisfying $x^3 + ax^2 + bx + c = 0$.

8. Prove or disprove: If G is a group and $H \leq G$ is any subgroup, then there is a one-to-one and onto function $f : G/H \to H\backslash G$. [Note: G is not assumed to be finite.]

9. Prove or disprove: If G is a group, $H \leq G$ is a subgroup and $\#G/H = 2$, then $H \triangleleft G$.

10. What is the exponent of S_8?

11. Is there a subgroup of $S_5 \times \mathbb{R}$ which is isomorphic to $\mathbb{Z}_5^2$? If so, exhibit such a group. If not, prove that it cannot exist.

12. Let $F := \mathcal{C}([0,1])$ be the set of continuous real-valued functions of the interval $[0,1]$. F is a ring when we define $(f+g)(x) := f(x)+g(x)$ and $(f \cdot g)(x) := f(x)g(x)$.

Let $I : F \to \mathbb{R}$ be defined by $I(f) := \int_0^1 f(x)x^2 dx$. Is I is a homomorphism of *rings*? Is it a homomorphism of additive *groups*?

13. Prove or disprove: If G is an abelian group and $n \in \mathbb{Z}_+$ is any positive integer, then $nG := \{g \in G \mid (\exists h \in G) g = nh := \overbrace{h + \cdots + h}^{n \text{ times}}\}$ is a normal subgroup and $G/nG \cong \mathbb{Z}_n$.

14. What is the multiplicative inverse of 13 in $\mathbb{Z}_{19}$?

15. Prove or disprove: For every positive integer $a < 223$, there is an integer x for which the remainder of $129x$ upon division by 223 is a.