MATH 113: ABSTRACT ALGEBRA
PRACTICE PROBLEMS FOR MIDTERM 1

1. Show that if $(G, \cdot)$ is a group of order 9, then G is abelian.

2. Let $(G, \cdot)$ be a group and X any set. Let F be the set of functions with domain X and range G. Define a binary operation $*$ on F by $(f * g)(x) := f(x) \cdot g(x)$. Is $(F, *)$ a group? If so, prove that it is. If not, give an axiom which is violated and prove that this is so.

3. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 4 & 3 & 2 & 7 & 6 & 1 & 5 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 2 & 7 & 8 & 1 & 6 & 4 \end{pmatrix}$

a. Write τ as a product of cycles.
b. Write σ as a product of transpostions.
c. Compute $\sigma\tau$ and $\tau\sigma$.
d. What is the order of σ? of $\sigma\tau$?

4. How many generators does the group $\mathbb{Z}_{225}$ have?

5. Let $G := [0, 1)$ be the set of real numbers x with $0 \leq x < 1$. Define an operation $*$ on G by

$$x * y := \begin{cases} x + y \text{ if } x + y < 1 \text{ and} \\ x + y - 1 \text{ if } x + y \geq 1 \end{cases}$$

Is $(G, *)$ a group? If so, prove that it is. If not, demonstrate how some axiom is violated.

6. Prove or disprove: Every associative binary operation on a set with two elements is commutative.

7. Complete the following table to form a multiplication table for a group (if possible) and explain why the resulting multiplication gives a group, or demonstrate that no such completion is possible.

$*$	e	a	b	c	d	f
e	e	a	b	c	d	f
a		e				
b			e			
c				e		
d					e	
f						e

8. Let $G := \mathbb{R}_+$ be the set of positive real numbers and let $\cdot$ be the usual multiplication operation. Is the function $x \mapsto x^2$ an isomorphism of G with itself? If so, prove so. If not, demonstrate that it is not.

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9. Prove or disprove: the set $\mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\}$ is a subgroup of $(\mathbb{C}, +)$.

10. Let $G \leq S_n$ be a subgroup of the symmetric group on n letters. Show that either every permutation in G is even or exactly half of the permutations in G are even.

11. Find a subgroup of S_5 which is isomorphic to the Klein group V.

12. Prove or disprove: every group of order 11 is commutative.

13. How many subgroups does S_4 have? Prove that your answer is correct.

14. Prove or disprove: If $(G, *)$ is a group and for every pair of elements $(a * b)^6 = a^6 * b^6$, then G is commutative.

15. Prove or disprove: If G is a group and $H \leq G$ and $K \leq G$ are two subgroups, then $(H \cap K) \leq G$.

16. Prove or disprove: If G is a finite group and some element of G has order equal to the size of G, then G is cyclic.

17. Consider the function $\sigma : \{0, \ldots, 15\} \to \{0, \ldots, 15\}$ defined by

$$x \mapsto \begin{cases} x + 4 \text{ if } x < 12 \\ x - 12 \text{ if } x \geq 12 \end{cases}$$

Show that σ is a permutation and describe its orbits.

18. Let G be the set of all permutations of $\mathbb{R}$ which move at most finitely many points. That is, $\sigma : \mathbb{R} \to \mathbb{R}$ belongs to G just in case $\sigma \in S_{\mathbb{R}}$ and $\{r \in \mathbb{R} : \sigma(r) \neq r\}$ is finite. Prove or disprove: G is a group under composition.

19. Let G be the set of 2×2 matrices having integer entries and a nonzero determinant. Prove or disprove: G is a group under matrix multiplication.

20. Let $(G, *)$ be a group and $a \in G$. Suppose that $a * a = a$. Prove or disprove: a must be the identity element.