# Math 225A – Model Theory

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## General Information

These notes are based on a course in *Metamathematics* taught by Professor Thomas Scanlon at UC Berkeley in the Autumn of 2013. The course will focus on Model Theory and the course book is Hodges' *a shorter model theory*.

As with any such notes, these may contain errors and typos. I take full responsibility for such occurences. If you find any errors or typos (no matter how trivial!) please let me know at mps@berkeley.edu.

## Lecture 18

## Stone Space

Let us recall the definition of $S_n(B)$ from last time.

**Definition.** Given a $\tau$-structure $\mathfrak{B}$, $A \subseteq \text{dom}(\mathfrak{B})$ and $n \in \omega$ then the **Stone space of $\mathfrak{B}$ over** $A$ is

$$S_n(A) := \{p : p \text{ is an } n\text{-type over } A\}$$

The basic open sets in $S_n(A)$ are of the form

$$(\varphi) := \{p \in S_n(A) : \varphi \in p\} \qquad \text{for } \varphi \in \mathscr{L}(\tau_{A,x_1,\ldots,x_n})$$

*Remark.* The spaces $S_n(A)$ are also called **type spaces**.

A topological space $X$ is **totally disconnected** if for all distinct elements $a$ and $b$ of $X$ there exists an open partition $U, V$ such that $a \in U$ and $b \in V$.

**Proposition.** *$S_n(A)$ is a totally disconnected compact space.*

*Proof.* For *totally disconnectedness*: Take $p \neq q$ from $S_n(A)$. Then there is some $\varphi$ in the symmetric difference of $p$ and $q$. Suppose $\varphi \in p \setminus q$. Then $p \in (\varphi)$ and $q \in (\neg\varphi)$ since types are complete. Now $S_n(A) = (\varphi) \cup (\neg\varphi)$ and $(\varphi) \cap (\neg\varphi) = \emptyset$. So $S_n(A)$ is totally disconnected.

Now for *compactness*: Suppose $\mathcal{U}$ is an open cover of $S_n(A)$. We may assume that $\mathcal{U}$ consists of basic open sets, i.e. there is some set $\Phi$ of formulae in $\mathscr{L}(\tau_{A,x_1,\ldots,x_n})$ such that $\mathcal{U} = \{(\varphi) : \varphi \in \Phi\}$.

Suppose towards contradiction that there is no finite subcover of $\mathcal{U}$ exists. Consider the theory

$$\mathrm{T} := \mathrm{Th}(\mathfrak{B}_A) \cup \{\neg\varphi(\bar{x}) : \varphi \in \Phi\}.$$

We claim that T is satisfiable. If not then by compactness there is some finite $\Phi_0 \subseteq \Phi$ such that

$$\mathrm{Th}(\mathfrak{B}_A) \cup \{\neg\varphi : \varphi \in \Phi_0\}$$

would be inconsistent. I.e.

$$\mathrm{Th}(\mathfrak{B}_A) \vdash \neg \bigwedge_{\varphi \in \Phi_0} \neg\varphi(\bar{x})$$

which implies

$$\mathrm{Th}(\mathfrak{B}_A) \vdash \bigvee_{\varphi \in \Phi_0} \varphi(\bar{x})$$

which is true if and only if

$$\mathfrak{B}_A \models \forall \bar{x} \bigvee_{\varphi \in \Phi_0} \varphi(\bar{x})$$

Then for any type $p$ in $S_n(A)$ since $p \supseteq \mathrm{Th}(\mathfrak{B}_A)$ we must have

$$p \vdash \bigvee_{\varphi \in \Phi_0} \varphi(\bar{x}).$$

Now since $p$ is complete it must satisfy one of the $\varphi(x)$, i.e.

$$p \vdash \varphi(\bar{x})$$

for some $\varphi(\bar{x}) \in \Phi_0$. Another way of saying this is that $p \in (\varphi)$. But then $p$ models $\mathrm{Th}(\mathfrak{B}_A) \cup \{\neg\varphi : \varphi \in \Phi_0\}$ contrary to the assumption that this theory is inconsistent.

Now applying compactness to the theory T we get a model $(\mathfrak{C}, \bar{b}) \models \mathrm{T}$. Letting $q = \mathrm{tp}(\bar{b}/A)$ then $q \supseteq \mathrm{T}$, so $q \notin \bigcup_{\varphi \in \Phi}(\varphi)$, but this is a contradiction since

$$q \in S_n(A) \setminus \bigcup_{\varphi \in \Phi} (\varphi).$$

which was assumed empty. □

*Remark.* In particular $S_n(A)$ is Hausdorff. Also note that the basic open sets $(\varphi)$ are clopen.

*Remark.* Another way of showing that $S_n(A)$ is compact would be to use Tychonoff's theorem (the product of compact spaces is compact). Then one would consider the map

$$S_n(A) \longrightarrow \prod_{\varphi \in \mathscr{L}(\tau_{A,\bar{x}})} \{0, 1\}$$

which sends $p \in S_n(A)$ to it's characteristic function,

$$\chi_p(\varphi) = \begin{cases} 0 & \text{if } \varphi \notin p \\ 1 & \text{if } \varphi \in p \end{cases}$$

Giving $\{0,1\}$ the discrete topology, and $\prod\{0,1\}$ the product topology we get, by Tychonoff's theorem that the product is compact. The map above is continuous, and injective. Furthermore the image is closed (hence compact) and the map is actually a homeomorphism onto it's image.

*Notation* (temporary). If the signature is ambiguous then we denote by $S_n^\tau(A)$ the space of $n$-types of $\mathfrak{B}$ over $A \subseteq \mathfrak{B}$ where $\mathfrak{B}$ is a $\tau$-structure.

Consider $\tau \subseteq \tau'$, an extension of signatures $\mathfrak{B}'$ a $\tau'$-structure and $A \subseteq B' := \mathrm{dom}(\mathfrak{B}')$. Then there is a restriction map

$$|_\tau : S_n^{\tau'}(A) \longrightarrow S_n^\tau(A)$$

sending $p \in S_n^{\tau'}(A)$ to $p|_\tau := p \cap \mathscr{L}(\tau_{A,\bar{x}})$.

**Proposition.** *The above map is continuous and surjective.*

*Proof. surjective*: Let $q \in S_n^\tau(A)$. We claim that $q \cup \mathrm{eldiag}(\mathfrak{B}')$ is consistent. If not then there is some finite set $\Xi \subseteq \mathrm{eldiag}(\mathfrak{B}')$ and finite $Q \subseteq q$ such that $\Xi \cup Q$ is inconsistent. We have

$$\Xi = \{\xi_1(b_1), \ldots, \xi_m(b_m)\}, \qquad \xi_i \in \mathscr{L}(\tau'_A) \text{ and } b_i \text{ a tuple from } B$$

and

$$Q = \{\varphi_1(\bar{x}), \ldots, \varphi_l(\bar{x})\}, \qquad \varphi_i \in \mathscr{L}(\tau_A).$$

By padding we can assume all the tuples $b_i$ are the same. Further, by conjunction we may assume that $m = l = 1$ so

$$\Xi = \{\xi(b)\} \qquad \text{and} \qquad Q = \{\varphi(\bar{x})\}.$$

Since $\Xi \cup Q$ is inconsistent we have

$$\xi(b) \vdash \neg\varphi(\bar{x})$$

(where $\bar{x} = x_1, \ldots, x_n$ are new constants, not appearing in $\mathcal{C}_{\tau'_B}$) so have

$$\xi(b) \vdash \forall\bar{x}\neg\varphi(\bar{x}).$$

But $\forall\bar{x}\neg\varphi(\bar{x})$ is a *sentence* in $\mathscr{L}(\tau_A)$. Now $\mathrm{Th}(\mathfrak{B}'_A|_\tau) \subseteq \mathrm{Th}(\mathfrak{B}'_B)$ and so $\mathfrak{B}'_B \models \xi(b)$ so $\mathfrak{B}'_B \models \forall\bar{x}\neg\varphi(\bar{x})$, which implies that $\forall\bar{x}\neg\varphi(\bar{x})$ is in $\mathrm{Th}(\mathfrak{B}'_A|_\tau)$. Since $q \in S_n^\tau(A)$ we have $q \supseteq \mathrm{Th}(\mathfrak{B}'_A|_\tau)$. So $\forall\bar{x}\neg\varphi(\bar{x}) \in q$ but $q \vdash \varphi(\bar{x})$ which is a contradiction. Thus, by compactness, $q \cup \mathrm{eldiag}(\mathfrak{B}')$ is consistent.

Let $(\mathfrak{B}'', \bar{b}) \models \mathrm{eldiag}(\mathfrak{B}') \cup q$. Set $p = \mathrm{tp}^{\tau'}(b/A)$. Then $p|_\tau = q$. So the map is surjective.

*Continuity*: Let $U \subseteq S_n^\tau(A)$ be a basic open set, say $U = (\varphi)^\tau$ for some $\varphi \in \mathscr{L}(\tau_{A,\bar{x}})$. Then $(-)|_\tau^{-1}(U) = (\varphi)^{\tau'}$, which is also basic open. Thus the map is continuous. ∎

**Corollary.** *Given $A \subseteq B \subseteq C$ there is a continuous onto map $S_n(B) \longrightarrow S_n(A)$.* $S$

So restriction of parameters defines a surjective continuous map.

*Example* (Finite type spaces). In the language of equality (i.e. $\tau = \emptyset$) we have $|S_1(\emptyset)| = 1$, $|S_2(\emptyset)| = 2$ and $|S_3(\emptyset)| = 5$.

*Example* (Countable type spaces). Let $\tau = \{E\}$ where $E$ is a binary relation symbol. Let the $\tau$-structure $\mathfrak{A} = (A, E)$ be such that $E$ is an equivalence relation with exactly one equivalence class of size $n$ for each $n \in \omega$ and no other equivalence classes. Then we claim that $|S_1(\emptyset)| = \aleph_0$. We can find $\aleph_0$ distinct elements of $S_1(\emptyset)$ by considering formulae (with one free variable $x$) expressing the number of elements related to $x$. For instance we could let $\varphi_n(x)$ be $\exists^{=n} y \ (y \neq x) \wedge E(y, x)$. This shows that $|S_1(\emptyset)| \geq \aleph_0$. We will not prove the other inequality. One way to approach this would be to prove some form of quantifier-simplification and then check that there are $\leq \aleph_0$ types.

*Example* (Maximal type space). Let $\tau$ be the signature of ordered fields and let $\mathfrak{A} = (\mathbb{Q}, +, \cdot, <, 0, 1)$. Then we claim $|S_n(\emptyset)| = 2^{\aleph_0}$. This is because we can define all rational numbers, and using these we can define all cuts of $\mathbb{Q}$, these are all consistent, taking the completion of these we see that there are at least $2^{\aleph_0}$ distinct types. However there cannot be any more since the language is countable and each type is a subset of the language.

*Remark.* Could there be a countable language $\mathscr{L}$ where the number of types over the empty set lies strictly between $\aleph_0$ and $2^{\aleph_0}$? The negative answer is known as *Vaught's Conjecture.*

Given a signaure $\tau$ and a set $\Delta(\bar{x}; \bar{y})$ of $\tau$-formulae where $\bar{x}$ is a finite tuple of new variables and $\bar{y}$ is arbitrary. Given a $\tau$-structure $\mathfrak{A}$, $b \in A^n$, $C \subseteq A$ we define a $\Delta$-type, $\text{tp}^{\Delta}(b/C)$ as

$$\begin{aligned} \text{tp}^{\Delta}(b/C) := \quad & \{\delta(x, d) : \delta(x, y) \in \Delta(x, y), d \text{ from } C, \text{ and } \mathfrak{A} \models \delta(b, d)\} \\ & \cup \{\neg\delta(x, d) : \delta(x, y) \in \delta(x, y)\Delta(x, y), d \text{ from } C, \text{ and } \mathfrak{A} \models \neg\delta(b, d)\} \end{aligned}$$

Then we let the set of $\Delta$-types, $S_n^{\Delta}(C)$ be the set of all maximal consistent sets of formulae of the form $\delta(x, d) \vee \neg\delta(x, d)$ as $\delta$ ranges through $\Delta$ and $d$ ranges through $C$.

As before there is a natural restriction map

$$|_{\Delta} : S_n(C) \longrightarrow S_n^{\Delta}(C)$$

which is continuous and surjective.

*Remark* (Concerning stability theory). We have defined $\Delta$-types in the way that Shelah defines them. This is a more syntactic way. There are however some semantic properties that one would expect them to have which they do not have. There is a subtle fix that can be found in Pillay's book.

**Proposition.** *Let $\mathfrak{A}$ be a $\tau$-structure. Let $\Delta_n(x_1, \ldots, x_n, \bar{y})$ be a set of formulae. Suppose that for all $n$ the restriction map*

$$|_\Delta : S_n(\emptyset) \longrightarrow S_n^\Delta(\emptyset)$$

*is a bijection. Then $\Delta = \bigcup_n \Delta_n$ is an elimination set for $\mathfrak{A}$.*

*Proof.* Let $\varphi \in \mathscr{L}(\tau_{x_1,\ldots,x_n})$ be a $\tau$-formula. We must show that $\varphi$ is equivalent to a boolean combination of elements of $\Delta$. Consider the theory

$$\mathrm{T} := \mathrm{Th}_\tau(\mathfrak{A}) \ \cup\ \{\varphi(a) \wedge \neg\varphi(b)\} \ \cup\ \{\delta(a) \longleftrightarrow \delta(b) : \delta \in \Delta_n(\bar{x})\}.$$

where $a$ and $b$ are new $n$-tuples of constant symbols.

T must be inconsistent. For suppose $(\mathfrak{A}', a, b) \models \mathrm{T}$. Then $\mathrm{tp}(a) \neq \mathrm{tp}(b)$ while $\mathrm{tp}^\Delta(a) = \mathrm{tp}^\Delta(b)$. By hypothesis this cannot happen since $S_n(\emptyset)$ is in bijection with $S_n^\Delta(\emptyset)$.

So (by compactness) there is some finite part, of T that is inconsistent. I.e. there are some $\delta_1, \ldots, \delta_l \in \Delta$ such that

$$\mathrm{T}^* := \mathrm{Th}(\mathfrak{A}) \ \cup\ \{\varphi(a) \wedge \neg\varphi(b)\} \ \cup\ \{\delta_i(a) \longleftrightarrow \delta_i(b) : i \leq l\}$$

is inconsistent.

*Notation.* Recall some notation previously used: $\theta^1 := \theta$ and $\theta^{-1} := \neg\theta$.

For $s \in \{-1, 1\}^l$ let,

$$\Phi_s := \bigwedge_{i=1}^{l} \delta_i^{s(i)}$$

and

$$\Psi := \bigvee_{\{s\ :\ \mathfrak{A} \models\ \exists x \varphi(x) \wedge \Phi_s(x)\}} \Phi_s$$

Note that $\Psi$ is a boolean combination of elements of $\Delta$. Now we claim that $\Psi$ is equivalent to $\varphi$. Suppose $a$ from $\mathfrak{A}$ satisfies $\Psi$, i.e. $\mathfrak{A} \models \Psi(a)$. So for some $s \in \{-1, 1\}^l$ we have

$$\mathfrak{A} \models \Phi_s(a)$$

and by definition of $\Psi$ we have, for the same $s$ that

$$\mathfrak{A} \models \exists x \varphi(x) \wedge \Phi_s$$

Let $b$ be from $\mathfrak{A}$ such that

$$\mathfrak{A} \models \varphi(b) \wedge \Phi_s(b).$$

So we have $\mathfrak{A} \models \Phi_s(a) \wedge \Phi_s(b)$, thus for all $i \leq l$ we get

$$\mathfrak{A} \models \delta_i(a) \longleftrightarrow \delta_i(b)$$

Now, since $\mathrm{T}^*$ is inconsistent, it follows that

$$\mathfrak{A} \models \varphi(a) \longleftrightarrow \varphi(b)$$

and since $\mathfrak{A} \models \varphi(b)$ we finally have $\mathfrak{A} \models \varphi(a)$. □

This is a powerful technique for proving quantifier elimination. If you can show – by automorphism arguments or some sort of semantic analysis – that some set $\Delta$ of formulae is enough to distinguish all types, then it is also enough to distinguish all formulae. This is the way that one proves quantifier elimination for more complicated structures.