Math 225A – Model Theory

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General Information

These notes are based on a course in *Metamathematics* taught by Professor Thomas Scanlon at UC Berkeley in the Autumn of 2013. The course will focus on Model Theory and the course book is Hodges' *a shorter model theory*.

As with any such notes, these may contain errors and typos. I take full responsibility for such occurences. If you find any errors or typos (no matter how trivial!) please let me know at mps@berkeley.edu.

Lecture 17

Compactness Continued

The compactness theorem proved last time has many consequences and will be used constantly from now on.

For instance if some formula $\varphi(x, y)$ always defines a *finite* subset (in all structures) then in fact there is some bound on the size of the definable subsets. More precisely we have the following proposition.

Proposition. *Let* T *be a* τ*-theory and* $\varphi(x, y) \in \mathscr{L}(\tau)$ *(where* x *and* y *are tuples). If for all* $\mathfrak{A} \models \mathrm{T}$ *and* $a \in A$ *we have that* $\varphi(\mathfrak{A}, a)$ *is finite, then there is some* $k \in \omega$ *such that* $\mathrm{T} \vdash \forall y \exists^{\leq k} x\ \varphi(x, y)$.

Proof. We expand the signature by adding constants $\{c_i : i \in \omega\} \cup \{d\}$. Call this new signature τ'. Consider the τ'-theory

$$S := \mathrm{T} \cup \{\varphi(c_i, d) : i \in \omega\} \cup \{c_i \neq c_j : i < j\}.$$

Suppose, towards contradiction, that the proposition is false, i.e. there is no bound $k \in \omega$. Then we claim that S is finitely satisfiable. By compactness we then get a model $\mathfrak{B}'$ of S. Taking the reduct $\mathfrak{B}'|_\tau$ we will see that $\varphi(\mathfrak{B}, a)$ is infinite for some a.

So we must show that S is finitely satisfiable. Let $S_0 \subseteq S$ be finite. Then there is some $N \in \omega$ such that

$$S_0 \subseteq \mathrm{T} \cup \{\varphi(c_i, d) : i \leq N\} \cup \{c_i \neq c_j : i < j < N\}.$$

By assumption there is no $k \in \omega$ that bounds $\varphi(\mathfrak{A}, a)$, so there is some $\mathfrak{A} \models \mathrm{T}$ and a from A such that $|\varphi(\mathfrak{A}, a)| > N$. Let $b_0, \ldots, b_{N-1}$ be N distinct elements from $\varphi(\mathfrak{A}, a)$. Expand $\mathfrak{A}$ to $\mathfrak{A}'$ (a τ'-structure) by

$$d^{\mathfrak{A}'} := a$$

and

$$c_i^{\mathfrak{A}'} := \begin{cases} b_i & \text{if } i < N \\ a & \text{otherwise} \end{cases}$$

(the choice a for $c_i^{\mathfrak{A}'}$ where $i \geq N$ is arbitrary, we could choose which ever element we want). Now $\mathfrak{A}' \models S_0$ since $\mathfrak{A} = \mathfrak{A}'|_\tau \models \mathrm{T}$ and $\mathfrak{A} \models \varphi(b_i, a)$ for $i < N$ and $\mathfrak{A} \models c_i \neq c_j$ for $i < j < N$. Thus S_0 is satisfiable.

By compactness there is a model $\mathfrak{B}'$ of S. Let $b_i := c_i^{\mathfrak{B}'}$ and $a := d^{\mathfrak{B}'}$. Let $\mathfrak{B} := \mathfrak{B}'|_\tau$. Then $\mathfrak{B} \models \mathrm{T}$ and the infinitely many distinct b_i's are all in the set $\varphi(\mathfrak{B}, a)$ which is a contradiction. This completes the proof. □

This proposition demonstrates a weakness of first-order logic. First-order logic cannot tell the difference between "arbitrarily large but finite" and "infinite". If we want to say "finite" then we must say "finite and bounded by k" for some k. The contrapositive is also interesting, namely that if we have a first-order theory with arbitrarily large finite models, then there is an infinite model. Of course this makes essential use of the first-order setting.

Example. Can one deduce the existence of an infinite well-ordered set from the existence of arbitrarily large finite well-ordered sets?

The most immediate approach using the compactness theorem never gets off the ground since being a well-ordered set is not a first-order property! It is however a second-order property, but second-order logic is not compact.

Instead what we can do is look at all sentences satisfied by all of the well-orders $\mathfrak{A}_n := (\{0, 1, \ldots, n-1\}, <)$. I.e. let T be the theory

$$\mathrm{Th}(\{\mathfrak{A}_n : n \in \omega\}).$$

Question. What is T? In fact T is the theory of discrete linear order with first and last elements. Can you prove this?

Applying compactness to

$$\mathrm{T} \cup \{c_i \neq c_j : i \neq j, i, j \in \omega\}$$

to obtain a model $\mathfrak{A}$ of T which is infinite.

The infinite model $\mathfrak{A}$ will not be a well-order, however it will contain an infinite well-order. Indeed $\mathfrak{A}$ will have a first element, say b (bottom) and last element, say t (top). t will have predecessors $P^n(t)$ and b will have successors $S^n(b)$ for all $n \in \omega$. Since $\mathfrak{A}$ is a linear order $S^n(b) \neq P^m(t)$ for any n, m, thus we get an infinite descending chain $t > P(t) > P^2(t) > \ldots$.

However the subset $b < S(b) < S^2(b) < \ldots$ *is* an infinite well-order.

We mentioned that being a well-order is not first-order expressible. This has not actually been proven yet.

Proposition. *If* $(X,<)$ *is an infinite linear ordered set then there exists* $(Y,<)$ *such that* $(X,<) \equiv (Y,<)$ *and such that* $(Y,<)$ *is* not *well-ordered.*

Proof. First extend the signature by constants: $\tau' = \{<\} \cup \{c_i : i \in \omega\}$. Let T be the τ'-theory

$$\mathrm{Th}((X,<)) \cup \{c_i > c_j : i < j\}$$

(note the "reversal" of the ordering of the c_i's). Now T is finitely satisfiable [proof: by finding an appropriate finite subset of X (which we assumed was infinite) which serves to give is a finite decreasing chain]. By compactness there is some infinite model Y' of T. Taking the reduct back to τ we have $(Y,<) \equiv (X,<)$ and Y has an infinite descending chain. □

The compactness theorem is very strong. As an example of its usefulness consider Ax's Theorem (Problem 13. Sec. 5.1 of Hodges). A variant of this theorem is the following.

Theorem 1 (Ax)**.** *If* $f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ *is given by polynomials and* f *has prime order, then* f *has a fixed point.*

Types

Definition. Given a τ-structure $\mathfrak{A}$, $n \in \omega$, $\bar{a} \in A^n$ and $B \subseteq A$ then the **type of** $\bar{a}$ **over** B, $\mathrm{tp}^{\mathfrak{A}}(\bar{a}/B)$ is

$$\mathrm{Th}(\mathfrak{A}_B, \bar{a})$$

thought of in the language $\mathscr{L}(\tau_{B,x_1,\ldots,x_n})$ where the x_i's are constant symbols which must only be substituted with the a_i's.

Notation. If the structure $\mathfrak{A}$ is clear from context then we write $\mathrm{tp}(\bar{a}/B)$ instead of $\mathrm{tp}^{\mathfrak{A}}(\bar{a}/B)$. Also we sometimes omit the bar above a even if a is a tuple.

Informally the type of a over B is the set of all formulae (with parameters from B) which are true of a inside $\mathfrak{A}$. Concretely we have

$$\mathrm{tp}^{\mathfrak{A}}(a/B) = \{\varphi(x_1,\ldots,x_n;\bar{b}) : \mathfrak{A} \models \varphi(a,\bar{b}) \text{ with } \bar{b} \text{ from } B \text{ and } \varphi \text{ from } \mathscr{L}(\tau)\}.$$

More generally,

Definition. An n**-type** over B (relative to $\mathfrak{A}$) is a complete finitely satisfiable theory in $\mathscr{L}(\tau_{B,x_1,\ldots,x_n})$ extending $\mathrm{Th}(\mathfrak{A}_B)$.

Definition. Given an n-type p we say that $a \in A^n$ **realizes** p if $p = \mathrm{tp}(a/B)$. If there is such an element in A^n then we say that p is **realized** in $\mathfrak{A}$. If there is no such element then we say that $\mathfrak{A}$ **omits** p.

We can always find an elementary superstructure wherein a given type is realized:

Proposition. *If $p(x_1, \dots, x_n)$ is an n-type, then there is some $\mathfrak{A}'$ which is an elementary extension of $\mathfrak{A}$ and $a \in (A')^n$ such that $p = \mathrm{tp}^{\mathfrak{A}'}(a/B)$.*

Proof. We use compactness. Let

$$\mathrm{T} := p \cup \mathrm{Th}(\mathfrak{A}_A)$$

(recall that $\mathrm{Th}(\mathfrak{A}_A) = \mathrm{eldiag}(\mathfrak{A})$ by definition). We claim that T is finitely satisfiable. Let $\mathrm{T}_0 \subseteq \mathrm{T}$ be finite. Let

$$\mathrm{T}_0 \cap p = \{\varphi_1(\bar{x}), \dots, \varphi_l(\bar{x})\}$$

We will show that $(\mathrm{T}_0 \cap p) \cup \mathrm{Th}(\mathfrak{A}_A)$ has a model by showing that there exists $\bar{a} \in A^n$ such that $(\mathfrak{A}_A, \bar{a}) \models (\mathrm{T}_0 \cap p) \cup \mathrm{Th}(\mathfrak{A}_A)$. So we want $a_1, \dots, a_n \in A$ such that

$$\mathfrak{A}_A \models \bigwedge_{i=1}^{l} \varphi_i(a_1, \dots, a_n).$$

We can find this if and only if

$$\mathfrak{A}_B \models \exists \bar{x} \bigwedge_{i=1}^{l} \varphi_i(\bar{x})$$

(remember that the φ_i's only involve parameters from B) which is true if and only if

$$\exists \bar{x} \bigwedge \varphi_i(\bar{x}) \in \mathrm{Th}(\mathfrak{A}_B) \subseteq p$$

which is true since p is complete.

So by compactness there is some $\mathfrak{A}' \models \mathrm{T}$. So $\mathfrak{A}' \models \mathrm{Th}(\mathfrak{A}_A)$ and so $\mathfrak{A} \preccurlyeq \mathfrak{A}'$. Furthermore

$$p = \mathrm{tp}(x_1^{\mathfrak{A}'}, \dots, x_n^{\mathfrak{A}'}/B)$$

since p is complete and $p \subseteq \mathrm{tp}(x_1^{\mathfrak{A}'}, \dots, x_n^{\mathfrak{A}'}/B)$, so p is realized in $\mathfrak{A}'$. □

Example. We give an example where a type is omitted. Let $\mathfrak{A} = (\mathbb{Q}, <)$ and let $B = \mathbb{Q}$. Let $C := \{q \in \mathbb{Q} : q < \sqrt{2}\}$ and $p(x)$ be the 1-type given by the complete extension of

$$\mathrm{Th}(\mathfrak{A}_{\mathbb{Q}}) \cup \{q < x | q \in C\} \cup \{x < q : q \in \mathbb{Q} \setminus C\}$$

Now p is finitely satisfiable since given any finite $p_0 \subseteq p$ we only mention finitely many

$$q_1 < \dots < q_n$$

and of these q_i's some are in C and some are not. Letting q_m be the maximal q_i contained in C then q_{m+1} is not in C. By density of $\mathbb{Q}$ there is some element r between q_m and q_{m-1} such that $\varphi(r)$ holds for all $\varphi \in p_0$. However p is not realized in $\mathfrak{A}$ since this would require $\sqrt{2} \in \mathbb{Q}$.

One way of realizing p in this case would be to let $\mathfrak{A}' = (\mathbb{Q} \cup \{\sqrt{2}\}, <)$ then $\mathfrak{A} \preccurlyeq \mathfrak{A}'$ and $\mathfrak{A}'$ realizes p.

It is worthwhile studying all types together as a topological space.

Definition. Given a τ-structure $\mathfrak{A}$ and $B \subseteq A$ and $n \in \omega$ the **Stone space** $S_n(B)$ (also denoted $S_X(B)$) is the set

$$\{p : p \text{ an } n\text{-type over } B \text{ relative to } \mathfrak{A}\}.$$

We topologies $S_n(B)$ by letting the basic open sets be

$$(\varphi) := \{p \in S_n(B) : \varphi \in p\}$$

for $\varphi \in \mathscr{L}(\tau_{B,x_1,\ldots,x_n})$.