Math 225A – Model Theory

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General Information

These notes are based on a course in *Metamathematics* taught by Professor Thomas Scanlon at UC Berkeley in the Autumn of 2013. The course will focus on Model Theory and the course book is Hodges' *a shorter model theory.*

As with any such notes, these may contain errors and typos. I take full responsibility for such occurences. If you find any errors or typos (no matter how trivial!) please let me know at mps@berkeley.edu.

Lecture 1

Model Theory is the study of the interrelation between structures and formal language instantiating the relations between semantics and syntax. We shall start by defining the structures. To define a structure we need the data of a *signature* and then an *interpretation* of the signature.

Definition. A **signature** τ consists of three (disjoint) sets $\mathcal{C}_\tau, \mathcal{F}_\tau, \mathcal{R}_\tau$ together with a function

$$\text{arity} : \mathcal{F}_\tau \cup \mathcal{R}_\tau \to \mathbb{Z}_+$$

The sets $\mathcal{C}_\tau, \mathcal{F}_\tau, \mathcal{R}_\tau$ will contain the constant symbols, function symbols and relation symbols, respectively. The arity function assigns to each function symbol and each relation symbol some positive integer thought of as the number of arguments that the function (respectively, the relation) takes. Note that we do not allow arities of functions and relations to be zero.

Definition. A **τ-structure** $\mathfrak{A}$ is given by a set A and interpretations of the elements of $\mathcal{C}_\tau \cup \mathcal{F}_\tau \cup \mathcal{R}_\tau$, i.e.:

- each $c \in \mathcal{C}_\tau$ is interpreted as an element $c^\mathfrak{A}$ in A.
- each $f \in \mathcal{F}_\tau$ is interpreted as a function $f^\mathfrak{A} : A^{\text{arity}(f)} \to A$.
- each $R \in \mathcal{R}_\tau$ is interpreted as a set $R^\mathfrak{A} \subseteq A^{\text{arity}(R)}$.

The set A is called the *domain of* $\mathfrak{A}$ and also denoted $\text{dom}(\mathfrak{A})$. We also use the notation $R(\mathfrak{A})$ for $R^\mathfrak{A}$ in anticipation of definable sets. For $a \in R^\mathfrak{A}$ we may also write $R(a)$.

Remark. In this course we do not require that A be nonempty!

Note that in order for $\emptyset$ to be a structure there can be no constant symbols (i.e. $\mathcal{C}_\tau = \emptyset$).

Remark. There is also a notion of a *sorted* signature, in which we would have another set $\mathcal{S}_\tau$ and in which "arity" would be replaced by giving the sort of each constant symbol, the domain and target of each function symbol and the field of each relation symbol. This is relevant in many situations for example when describing a vector space over a field (so we need two sorts: vectors and scalars) and in computer science.

Example (Groups). A group G may be regarded as a structure. The signature is in this case $\mathcal{C}_\tau = \{1\}$, $\mathcal{F}_\tau = \{\cdot\}$ and $\mathcal{R}_\tau = \emptyset$, and arity$(\cdot) = 2$.

As an interpretation we might let 1^G be the identity element of G and $\cdot^G : G \times G \to G$ the group multiplication.

Example (Graphs). A graph G is a triple (V, E, I) of vertices, edges and an incidence relation, such that for $e \in E$ and $v, w \in V$ we have $I(v, w, e)$ is $(v, w) \in e$ (i.e. if e is an edge between v and w).

There are two natural signatures to use that do give different notions of graphs as structures.

- Let τ be given by $\mathcal{C}_\tau = \emptyset$, $\mathcal{F}_\tau = \emptyset$ and $\mathcal{R}_\tau = \{V, E, I\}$ where arity$(V) =$ arity$(E) = 1$ and arity$(I) = 3$. With this signature we can now set dom$(G) = V \cup E$, $V^G = V$, $E^G = E$ and

$$I^G = \{(v, w, e) | (v, w) \in e\}.$$

- Let σ be the signature given by $\mathcal{C}_\sigma = \emptyset$, $\mathcal{F}_\sigma = \emptyset$ and $R_\sigma = \{E\}$ with arity$(E) = 2$. Now dom$(G) = V$ and

$$E^G = \{(v, w) \mid \exists e \in E \text{ such that } (v, w) \in e\}.$$

Now both signatures can be used to describe graphs but they are different. In the first case there can be multiple edges between the same vertices, while in the second there cannot. However in the first case we can have an edge $e \in E$ which is not connected to any vertices. So it makes a difference which language one uses!

We want to turn the collection of τ-structures into a category. For this we need morphisms.

Definition. A **homomorphism** $f : \mathfrak{A} \to \mathfrak{B}$ of τ-structures $\mathfrak{A}$ and $\mathfrak{B}$, is given by a function

$$f : \text{dom}(\mathfrak{A}) \to \text{dom}(\mathfrak{B})$$

which respects all the "extra structure". More precisely

- for all $c \in \mathcal{C}_\tau$ we have $f(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$

- for all $g \in \mathcal{F}_\tau$ (with say $n = \text{arity}(g)$) and $a_1, \ldots, a_n \in \text{dom}(\mathfrak{A})$ then

$$f(g^{\mathfrak{A}}(a_1, \ldots, a_n)) = g^{\mathfrak{B}}(f(a_1), \ldots, f(a_n)).$$

- for all $R \in \mathcal{R}_\tau$ (with say $n = \text{arity}(R)$) if $(a_1, \ldots, a_n) \in R^{\mathfrak{A}}$ then

$$(f(a_1), \ldots, f(a_n)) \in R^{\mathfrak{B}}.$$

Note that the notion of homomorphism depends on the choice of signature. For instance when defining homomorphisms of rings if we use a signature which has a constant symbol for the unit element then we get unit-preserving homomorphisms. If the signature does not have a constant symbol for the unit then homomorphisms of rings need not preserve the unit.

Proposition. *If $f : \mathfrak{A} \to \mathfrak{B}$ and $g : \mathfrak{B} \to \mathcal{C}$ are homomorphisms of τ-structures then $g \circ f$ is a homomorphism from $\mathfrak{A}$ to $\mathcal{C}$. Furthermore the identity map $1_A : A \to A$ gives a homomorphism of τ-structures $1_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}$.*

Proof. Exercise (purely formal). □

Thus, the collection of τ-structures together with homomorphisms between them form a category, called $\text{Str}(\tau)$.

Definition. For $\mathfrak{A}$ and $\mathfrak{B}$ τ-structures (with $\text{dom}(A) = A$ and $\text{dom}(\mathfrak{B}) = B$) then $\mathfrak{A}$ is a **substructure** of $\mathfrak{B}$, written $\mathfrak{A} \subseteq \mathfrak{B}$ if, $A \subseteq B$ and

- for all $c \in \mathcal{C}_\tau$, $c^{\mathfrak{A}} = c^{\mathfrak{B}}$.
- for all $f \in \mathcal{F}_\tau$, $f^{\mathfrak{A}} = f^{\mathfrak{B}}|_{A^n}$ where $n = \text{arity}(f)$.
- for all $R \in \mathcal{R}_\tau$, $R^{\mathfrak{A}} = R^{\mathfrak{B}} \cap A^n$ where $n = \text{arity}(R)$.

Proposition. *For τ-structures $\mathfrak{A}$ and $\mathfrak{B}$. If $\mathfrak{A} \subseteq \mathfrak{B}$ then the inclusion map ι : $\text{dom}(\mathfrak{A}) \to \text{dom}(\mathfrak{B})$ is a homomorphism.*

Proof. We have a function from $\text{dom}(\mathfrak{A})$ to $\text{dom}(\mathfrak{B})$. We check all three conditions

- let $c \in \mathcal{C}_\tau$ then $\iota(c^{\mathfrak{A}}) = c^{\mathfrak{A}} = c^{\mathfrak{B}}$ since $\mathfrak{A}$ is a substructure of $\mathfrak{B}$.
- ι commutes with all interpretations of the function symbols since $\iota = 1_B|_A$.
- if $(a_1, \ldots, a_n) \in R^{\mathfrak{A}}$ then by the substructure property $(\iota(a_1), \ldots, \iota(a_n)) = (a_1, \ldots, a_n) \in R^{\mathfrak{B}}$.

so ι is indeed a homomorphism. □

Warning!. The converse of the above proposition is not true in general! There exist τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ such that $\text{dom}(\mathfrak{A}) \subseteq \text{dom}(\mathfrak{B})$ and such that the inclusion map is a homomorphism and yet $\mathfrak{A}$ is *not* a substructure of $\mathfrak{B}$.

As an example, let τ be the signature given by $\mathcal{C}_\tau = \mathcal{F}_\tau = \emptyset$ and $\mathcal{R}_\tau = \{P\}$ with $\text{arity}(P) = 1$. Let $B = \mathbb{R}$ and $A = \mathbb{R}$ and consider these as τ-structures where $P^B = \mathbb{R}$ and $P^A = \emptyset$, respectively. Now the inclusion map $1_\mathbb{R} : A \to B$ is a homomorphism (we need only to check the condition on relation symbols, which is vacuous since $P^A = \emptyset$). However A is not a substructure of B since $P^A \neq P^B \cap A$.

Our notion of substructure is in some sense not the categorically correct notion. It is too restrictive. We want the homomorphisms (of τ-structures) to be the morphisms of $\text{Str}(\tau)$ but the above example shows that a subobject is not given by a monic morphism.

In some sense, some preservation of negation is built in to the definition of substructure, which is not built into the definition of homomorphism. That is, for $f : \mathfrak{A} \to \mathfrak{B}$ to be a homomorphism we require only that if some identity or relation holds in $\mathfrak{A}$ of some tuple $\overline{a}$, then the corresponding identity or relation holds of $\overline{f(a)}$. For $\mathfrak{A}$ to be a substructure of $\mathfrak{B}$, we require that not only is the inclusion map $\iota : \mathfrak{A} \hookrightarrow \mathfrak{B}$ a homomorphism, but if some identity or relation *fails* to hold in $\mathfrak{A}$ of some tuple $\overline{a}$, then the corresponding identity or relation fails to hold of a in $\mathfrak{B}$.

Proposition. *For τ-structures $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$. If $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{C}$ and $\text{dom}(\mathfrak{A}) = \text{dom}(\mathfrak{B})$ then $\mathfrak{A} = \mathfrak{B}$*

Proof. By hypothesis $\text{dom}(\mathfrak{A}) = \text{dom}(\mathfrak{B})$. For $c \in \mathfrak{C}_\tau$ we have $c^\mathfrak{A} = c^\mathfrak{C} = c^\mathfrak{B}$. Let $f \in \mathcal{F}_\tau$ and $R \in \mathcal{R}_\tau$ of arity n. We have $f^\mathfrak{A} = f^\mathfrak{C}|_{A^n} = f^\mathfrak{C}|_{B^n} = f^\mathfrak{B}$ by the substructure property. Finally $R^\mathfrak{A} = R^\mathfrak{C} \cap A^n = R^\mathfrak{C} \cap B^n = R^\mathfrak{B}$ again by the substructure property. □

So substructures are completely determined by their domain.

If $\{\mathfrak{B}_i\}_{i \in I}$ is a collection of substructures (with $B_i = \text{dom}(\mathfrak{B}_i)$) of $\mathfrak{A}$ then $\bigcap \mathfrak{B}_i$ will denote the τ-structure whose domain is $\bigcap \text{dom}(\mathfrak{B}_i)$ and where constants, functions and relations are interpreted on the intersection as before.

Proposition. *For $\mathfrak{A}$ a τ-structure and $X \subseteq \text{dom}(\mathfrak{A})$ there exists a smallest (with respect to inclusion) substructure $\langle X \rangle \subseteq \mathfrak{A}$ such that $X \subseteq \text{dom}(\langle X \rangle)$.*

Proof. We claim that if $\{\mathfrak{B}_i\}_{i \in I}$ is a collection of substructures of $\mathfrak{A}$ then $\bigcap \mathfrak{B}_i$ is a substructure of $\mathfrak{A}$.

If c is a constant symbol then $\forall i \in I$, $c^\mathfrak{A} \in B_i$ so $c^\mathfrak{A} \in \bigcap B_i$. For f a function symbol with arity n and $\bar{a} \in (\bigcap B_i)^n$ then $\forall j \in I$, $f^\mathfrak{A}(\bar{a}) \in B_j$ and so $f^\mathfrak{A}(\bar{a}) \in \bigcap B_i$. Thus $f^\mathfrak{A}|_{(\bigcap B_i)^n}$ is a function from $(\bigcap B_i)^n$ to $\bigcap B_i$. Likewise for R a relation symbol of arity n, define $R^{\bigcap B_i} := R^\mathfrak{A} \cap (\bigcap B_i)^n$. Now by definition $\bigcap \mathfrak{B}_i$ is a substructure of $\mathfrak{A}$.

With this in hand we now define the set of all substructures of $\mathfrak{A}$ that contain the subset X,

$$\mathscr{X} := \{\mathfrak{B} : \mathfrak{B} \subseteq \mathfrak{A} \text{ and } X \subseteq \text{dom}(\mathfrak{B})\}.$$

Note that $\mathfrak{A} \in \mathscr{X}$. By the above claim, $\bigcap \mathscr{X}$ is a substructure of $\mathfrak{A}$, and by definition $X \subseteq \text{dom}(\bigcap \mathscr{X})$. Furthermore if $X \subseteq \text{dom}(\mathfrak{B})$ for some substructure $\mathfrak{B}$ then since $\mathfrak{B} \in \mathscr{X}$ we have $\text{dom}(\bigcap \mathscr{X}) \subseteq \text{dom}(\mathfrak{B})$. Thus setting $\langle X \rangle = \bigcap \mathscr{X}$ we see that $\langle X \rangle$ is a substructure of $\mathfrak{A}$ whose domain contains X and whose domain is a subset of the domain of each substructure $\mathfrak{B}$ of $\mathfrak{A}$ which contains X. With the Squash Lemma below we conclude that $\langle X \rangle$ is in fact a substructure of each such $\mathfrak{B}$ and thereby complete the proof. □

Remark. The above proposition is true as stated since we allow structures with empty domains.

Lemma. *(The Squash Lemma) If* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ *are* τ*-structures and* $\text{dom}(\mathfrak{A}) \subseteq \text{dom}(\mathfrak{B}) \subseteq \text{dom}(\mathfrak{C})$*, and if both* $\mathfrak{A} \subseteq \mathfrak{C}$ *and* $\mathfrak{B} \subseteq \mathfrak{C}$*, then* $\mathfrak{A} \subseteq \mathfrak{B}$*.*

Proof. For $c \in \mathcal{C}_\tau$ we have $c^{\mathfrak{A}} = c^{\mathfrak{C}} = c^{\mathfrak{B}}$ by use of the given substructures. Similarly for function and relation symbols. □

The above construction of $\langle X \rangle$ doesn't actually show how to build $\langle X \rangle$. It works from above, since we know that there is some substructure containing X. Now we think of $\langle X \rangle$ as the substructure *generated* by X and we should be able to build $\langle X \rangle$ from below, simply using X.

In our attempt to build the substructure $\langle X \rangle$ we must first look at the constant symbols. For each $c \in \mathcal{C}_\tau$ if $c^{\mathfrak{A}} \notin X$ then we must add it. Futhermore for all function symbols and all tuples from X if f applied to these tuples is not in X then we must add these values. Thus we get a bigger set, and we can start over, and keep going until we finish. To make sense of this we introduce *terms*.

Definition. To a signature τ we have a set $\mathscr{T}(\tau)$ containing the **closed τ-terms**. $\mathscr{T}$ is given by recursion.

- $\mathscr{T}(\tau)$ contains all constant symbols
- if $f \in \mathcal{F}_\tau$ with $\text{arity}(f) = n$ and $t_1, \ldots, t_n \in \mathscr{T}(\tau)$ then $f(t_1, \ldots, t_n) \in \mathscr{T}(\tau)$.

Remark. The above definition of $\mathscr{T}(\tau)$ has two subtle issues. For one we did not specify exactly what a term is. Secondly it is not clear that the above recursive definition actually defines a set. To actually justify these details requires a bit of set theory.