Math 225A – Model Theory

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General Information

These notes are based on a course in *Metamathematics* taught by Professor Thomas Scanlon at UC Berkeley in the Autumn of 2013. The course will focus on Model Theory and the course book is Hodges' *a shorter model theory.*

As with any such notes, these may contain errors and typos. I take full responsibility for such occurences. If you find any errors or typos (no matter how trivial!) please let me know at mps@berkeley.edu.

Lecture 9

Skolemisation

Theorem 1 (Skolemisation Theorem)**.** *For any signature τ there exists a signature τ' ($= \tau^{skolem}$) and a τ'-theory T' ($= T^{skolem}$) such that*

- *T' has skolem functions*
- *Every τ-structure extends to a τ'-structure which models T'. I.e. the restriction map* $\mathrm{Mod}(T') \longrightarrow \mathrm{Str}(\tau)$ *is surjective.*

Proof. The basic idea is just to put the required Skolem functions into the signature. Of course just doing this for τ (say by extending τ to τ^+) doesn't work since there will be new formulas in the language $\mathscr{L}(\tau^+)$ which lack Skolem functions. To remedy this we construct a chain and take a union.

We construct an increasing sequence of signatures $\tau_0 \subseteq \tau_1 \subseteq \cdots$ and theories $T_0 \subseteq T_1 \subseteq \cdots$ (where T_n is an $\mathscr{L}(\tau_n)$-theory). Then let

$$\tau' := \bigcup_n \tau_n \qquad \text{and} \qquad T' := \bigcup_n T_n.$$

The construction is as follows.

- Let $\tau_0 := \tau$ and $T_0 := \emptyset$.
- At stage n define $\mathcal{C}_{\tau_{n+1}} := \mathcal{C}_{\tau_n}$, $\mathcal{R}_{\tau_{n+1}} := \mathcal{R}_{\tau_n}$, and

  $$\mathcal{F}_{\tau_{n+1}} := \mathcal{F}_{\tau_n} \cup \{f_{(\varphi,m)} : \varphi \in \mathscr{L}(\tau_n), \text{ with free variables amongst } x_0, \ldots, x_{m-1}\}.$$

  The theory T_{n+1} will be T_n together with

  $$\{\forall x_0, \ldots, x_{m-1}[\varphi(x_0, \ldots, x_{m-1}, f_{(\varphi,m)}(\bar{x})) \longleftrightarrow \exists y\ \varphi(\bar{x}, y)] \\ : \varphi \in \mathscr{L}(\tau_n) \text{ free variables in } x_0, \ldots, x_{m-1}\}$$

Now we claim that T' has Skolem functions. Indeed if $\varphi(\bar{x}, y) \in \mathscr{L}(\tau')$ with free variables amongst $\bar{x}, y$ then $\varphi(\bar{x}, y) \in \mathscr{L}(\tau_n)$ for some n. Now we constructed T_{n+1} to say that

$$\forall \bar{x}(\varphi(\bar{x}, f_{(\varphi,m)}(\bar{x})) \longleftrightarrow \exists y\ \varphi(\bar{x}, y)).$$

So T' does have a skolem function for φ.

Now we show the second claim, namely that the restriction map $\mathrm{Mod}(T') \longrightarrow \mathrm{Str}(\tau)$ is surjective. We show that if $\mathfrak{A}_n \in \mathrm{Mod}(T_n)$ then there exists $\mathfrak{A}_{n+1} \in \mathrm{Mod}(T_{n+1})$ such that $\mathfrak{A}_n = \mathfrak{A}_{n+1}|_{\tau_n}$; To find $\mathfrak{A}_{n+1}$ we basically need to show how to interpret the Skolem functions that entered at stage n.

Let $f_{\varphi,m} \in \mathcal{F}_{\tau_{n+1}}$ be a new function symbol in τ_{n+1}. To interpret $f_{\varphi,m}$ we shall need the *axiom of choice*:

For all sets X there exists a map $f : X \setminus \{\emptyset\} \to \bigcup X$ such that $\forall y \in X\ f(y) \in y$.

Let $X = \{\varphi(\bar{a}, \mathfrak{A}) : \bar{a} \in A^m\}$ and let g be a choice function for X as afforded by the axiom of choice. Then define

$$f_{\varphi,m}^{\mathfrak{A}_{n+1}}(\bar{a}) = \begin{cases} g(\varphi(\bar{a}, \mathfrak{A})) & \text{if } \varphi(\bar{a}, \mathfrak{A}) \neq \emptyset \\ a_0 & \text{otherwise} \end{cases}$$

Note that the second clause only happens when $\varphi(\bar{a}, \mathfrak{A})$ is empty, but then whatever $f_{\varphi,m}$ does to $\bar{a}$ doesn't matter. This interpretation makes $\mathfrak{A}_{n+1}$ into a τ_{n+1}-structure which models T_{n+1} and restricts back to $\mathfrak{A}_n$. $\square$

For a model $\mathfrak{B}$ of a theory T with Skolem function the notion of substructure and elementary substructure coincide! This follows as a Corollary to the following proposition.

Proposition. *If T is a τ-theory with Skolem functions then for every formula $\theta(\bar{x})$ with* at least one free variable*, there is a quantifier-free formula $\varphi(\bar{x})$ such that $T \models \forall \bar{x}(\theta(\bar{x}) \leftrightarrow \varphi(\bar{x}))$.*

Proof. We work by induction on the complexity of θ. The atomic case is immediate. Boolean combinations are also immediate. For the case $\theta(\bar{x})$ is $\exists y\ \psi(\bar{x}, y)$ then we have

$$T \models \forall \bar{x}(\theta(\bar{x}) \leftrightarrow \psi(\bar{x}, f_\psi))$$

since T has Skolem functions. By induction we may find an equivalent formula for ψ. $\square$

Corollary. *If T has Skolem functions and $\mathfrak{B} \models T$, then if $\mathfrak{A} \subseteq \mathfrak{B}$ then $\mathfrak{A} \preccurlyeq \mathfrak{B}$ provided $\mathfrak{A} \neq \emptyset$.*

Proof. Let $\mathfrak{A} \subseteq \mathfrak{B}$ be nonempty. We show $\mathfrak{A} \preccurlyeq \mathfrak{B}$. By the Tarski-Vaught criterion it suffices to check that for any formula $\varphi(\bar{x}, y)$ and $\bar{a}$ a tuple of elements from $\mathfrak{A}$ (which exists since $\mathfrak{A} \neq \emptyset$) then $\mathfrak{A} \models \exists y \; \varphi(\bar{a}, y))$ iff $\mathfrak{B} \models \exists y \; \varphi(\bar{a}, y)$. The forward direction is immediate. Suppose $\mathfrak{B} \models \exists y \; \varphi(\bar{a}, y)$ then since T has Skolem functions, $\mathfrak{B} \models \varphi(\bar{a}, f_\varphi(\bar{a}))$ and since $\varphi(\bar{a}, f_\varphi(\bar{a}))$ only involves parameters from $\mathfrak{A}$ we have $\mathfrak{A} \models \varphi(\bar{a}, f_\varphi(\bar{a}))$ which implies that $\mathfrak{A} \models \exists y \; \varphi(\bar{a}, y)$. □

So skolemisation gives a way of building elementarily equivalent substructures. As a corollary we get the full Downward Löwenheim-Skolem theorem.

Theorem 2 (Downward Löwenheim-Skolem theorem)**.** *Let $\mathscr{L}(\tau)$ be a first-order language, $\mathfrak{A}$ a τ-structure, X a set of elements of $A = \mathrm{dom}(\mathfrak{A})$ and λ a cardinal such that $|\mathscr{L}(\tau)| + |X| \leq \lambda \leq |\mathfrak{A}|$. Then $\mathfrak{A}$ has an elementary substructure $\mathfrak{B}$ of cardinality λ with $X \subseteq \mathrm{dom}(\mathfrak{B})$.*

Proof. We skolemise the empty τ-theory $T = \emptyset$ to get a τ^{skolem}-theory T^{skolem} and an extension of $\mathfrak{A}$ to a model $\mathfrak{A}^{skolem}$ of T^{skolem}. Let Y be a subset of A with $|Y| = \lambda$ and $X \subseteq Y$. Then Let $\mathfrak{B}'$ be the substructure generated by Y. Finally take the reduct $\mathfrak{B}$ of $\mathfrak{B}'$ to τ. Now $|\mathfrak{B}| \leq |Y| + |\mathscr{L}(\tau^{skolem})| = \lambda + |\mathscr{L}(\tau)| = \lambda = |Y| \leq |\mathfrak{B}|$. By the above corollary $\mathfrak{B}' \preccurlyeq \mathfrak{A}^{skolem}$ hence $\mathfrak{B} \preccurlyeq \mathfrak{A}$. □

Games

We will now discuss games for testing equivalence of structures. There are many different forms of games. Different forms of games will bring different notions of equivalence which correspond to different logics on the structures.

As a prelude we prove a famous theorem due to Cantor.

Theorem 3 (Cantor's Back-and-Forth Theorem)**.** *If $(A, \leq)$ and $(B, \leq)$ are nonempty countable dense linear orders without endpoints then they are isomorphic.*

Notation. The abbreviation "*DLO*" is commonly used for the theory of dense linear orders without endpoints.

Proof. Let $A = \{a_n : n \in \omega\}$ and $B = \{b_n : n \in \omega\}$ be some enumerations of A and B respectively. We shall construct an increasing sequence $\{f_n : n \in \omega\}$ of partial isomorphisms (i.e. f_n is an isomorphisms between its domain and codomain thought of as substructures) such that

1. $f_n \subseteq f_{n+1}$,
2. $\mathrm{dom}(f_n) \supseteq \{a_0, \ldots, a_{n-1}\}$ and $\mathrm{range}(f_n) \supseteq \{b_0, \ldots, b_{n-1}\}$,
3. and f_n is finite for all n.

We do this as follows. We let $f_0 := \emptyset$. At stage $n+1$ we want to extend f_n to ensure that $a_n \in A$ is in the domain and that b_n is in the range of f_{n+1}. There are four cases.

- If $a_n \in \mathrm{dom}(f_n)$ then $f_{n+\frac{1}{2}} = f_n$
- If $\forall a \in \mathrm{dom}(f_n)\ a_n < a$ then since $B \models DLO$ there exists $b' \in B$ such that $\forall b \in \mathrm{range}(f_n)\ b' < b$ (here we are using that $\mathrm{range}(f_n)$ is finite). Then set $f_{n+\frac{1}{2}}(a_n) = b'$.
- If $\forall a \in \mathrm{dom}(f_n)\ a < a_n$ then since $B \models DLO$ there exists $b' \in B$ such that $\forall b \in \mathrm{range}(f_n)\ b < b'$ (again since $\mathrm{range}(f_n)$ is finite). Set $f_{n+\frac{1}{2}}(a_n) = b'$.
- If there is $a, b \in \mathrm{dom}(f_n)$ with $a < b$ and $(a, b) \cap \mathrm{dom}(f_n) = \emptyset$ and $a < a_n < b$ then $\forall c \in \mathrm{range}(f_n)$ we have $\neg(f_n(a) < c < f_n(b))$ since f_n is an isomorphism. Now since B is dense there is some d such that $f_n(a) < d < f_n(b)$. Pick such a d and define $f_{n+\frac{1}{2}}(a_n) = d$.

This tells us how to map a_n forward. Now dual arguments show how to extend f_n^{-1} to $f_{n+\frac{1}{2}}^{-1}$ so that $f_{n+\frac{1}{2}}^{-1}$ is defined on b_n. Putting both directions together we get the maps f_{n+1} and f_{n+1}^{-1}.
The sequence $(f_n)_{n\in\omega}$ clearly satisfies the requirements 1. 2. and 3.. Now letting

$$f = \bigcup_n f_n$$

we get that f is an isomorphism between $A = \mathrm{dom}(f)$ and $B = \mathrm{range}(f)$. □

Remark. An alternative formulation of the theorem is that DLO is an $\aleph_0$-categorical theory.

The Ehrenfeucht-Fraïssé game

The proof of Cantor's theorem is an example of the *back and forth method.* We can formalize this argument in terms of a game namely the *Ehrenfeucht-Fraïssé game of length ω.*

There are two players; $\forall$ (Abelard) and $\exists$ (Heloise). Let γ be an ordinal. The **Ehrenfeucht-Fraïssé game** of length γ between τ-structures $\mathfrak{A}$ and $\mathfrak{B}$, denoted $\mathrm{EF}_\gamma(\mathfrak{A}, \mathfrak{B})$, has γ moves. At move α, $\forall$ picks an element from either $\mathfrak{A}$ or $\mathfrak{B}$. $\exists$ responds with an element from the other model. A play of the $\mathrm{EF}_\gamma(\mathfrak{A}, \mathfrak{B})$-game is a γ-tuple $(a_\alpha, b_\alpha)_{\alpha<\gamma}$ where $a_\alpha \in A$ and $b_\alpha \in B$.
Player $\exists$ wins the play $(a_\alpha, b_\alpha)_{\alpha<\gamma}$ if the map $a_\alpha \longmapsto b_\alpha$ has the property that for every for atomic formula $\varphi(\bar{x})$ we have

$$\mathfrak{A} \models \varphi(\bar{a}) \qquad \Longleftrightarrow \qquad \mathfrak{B} \models \varphi(\bar{b})$$

where $\bar{a}$ is a tuple from $(a_\alpha)_{\alpha<\gamma}$ and $\bar{b}$ is the image of $\bar{a}$ under the map $a_\alpha \longmapsto b_\alpha$. A **winning strategy** for $\mathrm{EF}_\gamma(\mathfrak{A}, \mathfrak{B})$ is a function from the set of partial plays to plays up through stage α together with $\forall$'s play are stage α, which returns a play for $\exists$, such that if $\exists$ follows this function then she always wins.

Definition. We say that $\mathfrak{A}$ is **γ-equivalent** to $\mathfrak{B}$, written $\mathfrak{A} \sim_\gamma \mathfrak{B}$, if $\exists$ has a winning strategy for $\mathrm{EF}_\gamma(\mathfrak{A}, \mathfrak{B})$.

Remark. Note that if $\mathfrak{A} \cong \mathfrak{B}$ then $\mathfrak{A} \sim_\gamma \mathfrak{B}$ for any ordinal γ. The winning strategy is given simply by the isomorphism.

Remark. $\mathfrak{A} \sim_0 \mathfrak{B}$ if and only if for all atomic *sentences* φ we have $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$.

Example. $(\mathbb{Q}, <) \sim_\omega (\mathbb{R}, <)$. This follows from the Back and Forth method demonstrated in the proof of Cantors theorem.

Example. $(\mathbb{Q}, <) \not\sim_{\omega+1} (\mathbb{R}, <)$. To see this we must show that $\forall$ can force a win in the $\mathrm{EF}_{\omega+1}((\mathbb{Q}, <), (\mathbb{R}, <))$-game. To do this $\forall$ may start with an enumeration $(q_n)_{n<\omega}$ of $\mathbb{Q}$. At each stage $n < \omega + 1$ in the game, $\forall$ picks an element of $\mathbb{R}$ corresponding to the rational number q_n sitting inside of $\mathbb{R}$. Then $\exists$ must always pick elements from $\mathbb{Q}$. Now at the ω'th play $\forall$ picks some irrational element of $\mathbb{R}$. $\exists$ must now pick one of its previous choices from $\mathbb{Q}$ and looses the game since the resulting function will not be an isomorphism.