Math 225A – Model Theory

Speirs, Martin

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General Information

These notes are based on a course in *Metamathematics* taught by Professor Thomas Scanlon at UC Berkeley in the Autumn of 2013. The course will focus on Model Theory and the course book is Hodges' *a shorter model theory*.

As with any such notes, these may contain errors and typos. I take full responsibility for such occurences. If you find any errors or typos (no matter how trivial!) please let me know at mps@berkeley.edu.

Lecture 16

Compactness

We shall now prove the compactness theorem. The proof we give is by the so-called Henkin construction. Later in the course we will give a second proof making use of ultrafilters.

Definition. A theory T is said to be **finitely satisfiable** if whenever $\mathrm{T}_0 \subseteq \mathrm{T}$ is a *finite* subset of T there exists a model $\mathfrak{A}$ of T_0.

Theorem 1 (The Compactness Theorem)**.** *Let τ be a signature and* T *a τ-theory. If* T *is finitely satisfiable then* T *is satisfiable, i.e. then there exists a model $\mathfrak{A} \models \mathrm{T}$.*

Remark. Compactness, as proved here, is a property of *first-order logic*, i.e. the sentences of T are assumed to come from the first-order language $\mathscr{L}_{\omega\omega}(\tau)$.

We will use the technique of using the language itself to build a structure satisfying T.

We first prove some lemma's allowing us to reduce the problem of finding models of T to that of finding models of a certain nice extension of T.

Definition. A τ-theory T **has Henkin constants** if for each formula $\varphi(x) \in \mathscr{L}(\tau)$ with one free variable x, there is a constant symbol $c \in \mathcal{C}_\tau$ such that

$$\mathrm{T} \vdash \exists x\ \varphi(x) \longleftrightarrow \varphi(c).$$

Henkin constants act ass witnesses to all existential sentences, iff they are implied by T.

Proposition. *For any signature τ there is a signature τ^{Hen} expressible as $\bigcup_{i=0}^{\infty} \tau_{(i)}^{\mathrm{Hen}}$ and a theory $\mathrm{T}^{\mathrm{Hen}}$ in $\mathscr{L}(\tau^{\mathrm{Hen}})$ such that*

1.) *τ^{Hen} is an expansion by constants of τ.*

2.) $\mathrm{T}^{\mathrm{Hen}}$ *has Henkin constants.*

3.) *For each* $\mathfrak{A} \in \mathrm{Str}(\tau)$ *there is some non-zero* $\mathfrak{A}' \in \mathrm{Str}(\tau^{\mathrm{Hen}})$ *such that* $\mathfrak{A}' \models \mathrm{T}^{\mathrm{Hen}}$ *and* $\mathfrak{A}$ *is the* τ*-reduct of* $\mathfrak{A}'$*.*

Proof. We define $\tau^{\mathrm{Hen}}_{(n)}$ and $\mathrm{T}^{\mathrm{Hen}}_n$ recursively in n. Let $\tau^{\mathrm{Hen}}_{(0)} := \tau$ and $\mathrm{T}^{\mathrm{Hen}}_0 := \emptyset$. At stage n we expand $\tau^{\mathrm{Hen}}_{(n)}$ by adding constants only, indeed let

$$\mathcal{C}_{\tau^{\mathrm{Hen}}_{(n+1)}} := \mathcal{C}_{\tau^{\mathrm{Hen}}_{(n)}} \cup \{c_\varphi : \varphi \text{ is in } \mathscr{L}(\tau^{\mathrm{Hen}}_{(n)}) \text{ with exactly 1 free variable}\}.$$

We also expand $\mathrm{T}^{\mathrm{Hen}}_n$ to state that the new constants c_φ act as witnesses, i.e. let

$$\mathrm{T}^{\mathrm{Hen}}_{n+1} := \mathrm{T}^{\mathrm{Hen}}_n \cup \{\exists x \varphi(x) \longleftrightarrow \varphi(c_\varphi) : \varphi \in \mathscr{L}(\tau^{\mathrm{Hen}}_{(n)}) \text{ with exactly 1 free variable}\}.$$

Then we define

$$\tau^{\mathrm{Hen}} := \bigcup_n \tau^{\mathrm{Hen}}_{(n)} \qquad \text{and} \qquad \mathrm{T}^{\mathrm{Hen}} := \bigcup_n T^{\mathrm{Hen}}_n.$$

Clearly $\mathrm{T}^{\mathrm{Hen}}$ has Henkin constants and τ^{Hen} is an expansion of τ by constants. This takes care of *1.)* and *2.)* in the proposition.

Now we show that τ^{Hen} and $\mathrm{T}^{\mathrm{Hen}}$ satisfy property *3.)*. Let $\mathfrak{A}$ be non-empty τ-structure [1]. We will find $\mathfrak{A}_{(n)} \in \mathrm{Str}(\tau^{\mathrm{Hen}}_{(n)})$ such that $\mathfrak{A}_{(0)} = \mathfrak{A}$ and $\mathfrak{A}_{(n)} = \mathfrak{A}_{(n+1)}|_{\tau^{\mathrm{Hen}}_{(n)}}$ and such that $\mathfrak{A}_{(n)} \models \mathrm{T}^{\mathrm{Hen}}_n$.

For $n = 0$ let $\mathfrak{A}_{(0)} := \mathfrak{A}$, then have $\mathfrak{A}_{(0)} \models \mathrm{T}^{\mathrm{Hen}}_0 = \emptyset$.

Given $\mathfrak{A}_{(n)}$, for each $\varphi \in \mathscr{L}(\tau^{\mathrm{Hen}}_{(n)})$ with free variable x if $\mathfrak{A}_{(n)} \models \exists x \varphi$ let $a_\varphi \in \varphi(\mathfrak{A}_{(n)})$, if $\mathfrak{A}_{(n)} \models \neg\exists x \varphi(x)$ then let a_φ be arbitrary. *Here we have used the axiom of choice to pick the witnesses* a_φ. We interpret

$$c_\varphi^{\mathfrak{A}_{(n+1)}} := a_\varphi.$$

This ensures that $\mathfrak{A}_{(n+1)} \models \mathrm{T}^{\mathrm{Hen}}_{n+1}$.

Finally let $\mathfrak{A}'$ be the unique τ^{Hen} structure with $\mathfrak{A}'|_{\tau^{\mathrm{Hen}}_{(n)}} = \mathfrak{A}_{(n)}$. $\mathfrak{A}'$ is the desired structure. □

Corollary. *If* T *is a finitely satisfiable* τ*-theory then* $\mathrm{T} \cup \mathrm{T}^{\mathrm{Hen}}$ *is a finitely satisfiable* τ^{Hen}*-theory.*

Proof. Let $S' \subseteq \mathrm{T} \cup \mathrm{T}^{\mathrm{Hen}}$ be finite. Then $S := S' \cap \mathrm{T}$ is finite. By hypothesis there is some $\mathfrak{A}$ such that $\mathfrak{A} \models S$. By the proposition there is some expansion $\mathfrak{A}' \in \mathrm{Str}(\tau^{\mathrm{Hen}})$ such that $\mathfrak{A}' \models \mathrm{T}^{\mathrm{Hen}}$ and such that $\mathfrak{A}'|_\tau = \mathfrak{A}$. This then implies that $\mathfrak{A}' \models S$. But since $S' \subseteq S \cup \mathrm{T}^{\mathrm{Hen}}$ we see that $\mathfrak{A}' \models S'$. □

[1] If $\mathfrak{A}$ is empty then condition *3.)* is vacuously satisfied since it is never the case that $\exists \varphi(x)$ is true.

Thus given any finitely satisfiable theory T we can canonically expand the language and the theory to get a finitely satisfiable theory $\mathrm{T}^{\mathrm{Hen}}$ *which has Henkin constants*. So in proving the compactness theorem it will suffice to consider only the case where T has Henkin constants.

We shall make one more reduction of the problem before proving the compactness theorem. This time we show how to extend to complete theories. Recall that we say a theory T is **complete** if for every sentence φ either $\varphi \in \mathrm{T}$ or $\neg\varphi \in T$.

Proposition. *If* T *is a finitely satiable* τ*-theory then there is a* complete *extension* $\widetilde{\mathrm{T}} \supseteq \mathrm{T}$ *which is still finitely satisfiable.*

Proof. We use Zorn's lemma to find a maximal finitely satisfiable extension of T and then argue by maximality that this extension must be complete.

Indeed let $\mathcal{P}$ be the partially ordered (by inclusion) set of extensions $\mathrm{T}' \supseteq \mathrm{T}$ which are finitely satisfiable. $\mathcal{P}$ is non-empty since $\mathrm{T} \in \mathcal{P}$. Taking a non-empty chain in $\mathcal{P}$ then the union of the chain is also an element of $\mathcal{P}$ since any finite subset of the union is contain in one of the elements of the union and therefore satisfiable. Therefore by Zorn's lemma there is a maximal finitely satisfiable extension $\widetilde{\mathrm{T}} \supseteq \mathrm{T}$.

Now we claim that $\widetilde{\mathrm{T}}$ is complete. Suppose by way of contradiction that φ is a $\mathscr{L}(\tau)$-sentence and such that both φ and $\neg\varphi$ are *not* in $\widetilde{\mathrm{T}}$. Then both

$$\widetilde{\mathrm{T}} \cup \{\varphi\} \supsetneq \widetilde{\mathrm{T}} \supseteq \mathrm{T}$$

and

$$\widetilde{\mathrm{T}} \cup \{\neg\varphi\} \supsetneq \widetilde{\mathrm{T}} \supseteq \mathrm{T}$$

so neither $\widetilde{\mathrm{T}} \cup \{\varphi\}$ nor $\widetilde{\mathrm{T}} \cup \{\neg\varphi\}$ are elements of $\mathcal{P}$. Since they both contain T, the only way they can avoid being in $\mathcal{P}$ is if they are not finitely satisfiable. So there is some $U, V \subseteq \widetilde{\mathrm{T}}$ such that $U \cup \{\varphi\}$ and $V \cup \{\neg\varphi\}$ are not satisfiable. But now $U \cup V \subseteq \widetilde{\mathrm{T}}$ is finite hence satisfiable. Let $\mathfrak{A}$ be a model of $U \cup V$. Now either φ or $\neg\varphi$ holds in $\mathfrak{A}$, either way we have a contradiction since one of $U \cup \{\varphi\}$ and $V \cup \{\neg\varphi\}$ will be satisfied by $\mathfrak{A}$. □

So starting with any finitely satisfiable theory T in any signature τ we can expand the signature and the theory to get a $\tau' \supseteq \tau$ and $\mathrm{T}' \supseteq \mathrm{T}$ which has Henkin constants. We can now extend further to another τ'-theory $\mathrm{T}'' \supseteq \mathrm{T}'$ which is complete. Then T'' still has Henkin constants and is also complete. If T'' is satisfiable then we can take a reduction back to τ to see that T is also satisfiable.

Proposition. *If* T *is a finitely satisfiable theory with Henkin constants, then there exists a model* $\mathfrak{A}$ *of* T*. In fact we may take* $\mathrm{dom}(\mathfrak{A})$ *to be* $\{c^{\mathfrak{A}} : c \in \mathcal{C}_\tau\}$.

Proof. Let $\mathcal{C} := \mathcal{C}_\tau$ be the set of constant symbols. We define a relation $\sim$ on $\mathcal{C}$ by $c \sim d$ iff the sentence $c{=}d$ is in T. We will show that $\sim$ is an equivalence relation and then let $\mathrm{dom}(\mathfrak{A})$ be the set of equivalence classes.

First, to see that $\sim$ is an equivalence relation we check the three axioms. They all follow the same pattern so let us just show the reflexivity: Let $c \in \mathcal{C}$. Since T is complete either $c{=}c$ or $c{\neq}c$ is in T. But T is also finitely satisfiable and since there is no model satisfying $c{\neq}c$ we must have $c{=}c \in \mathrm{T}$, and so $c \sim c$.

We now define a τ-structure $\mathfrak{A}$ with domain $\mathcal{C}/\sim$. For $c \in \mathcal{C}_\tau$ let $c^{\mathfrak{A}} := [c]_\sim$. For $f \in \mathcal{F}_\tau$ of arity n, then given $c_0, \dots, c_{n-1} \in \mathcal{C}_\tau$ then

$$f^{\mathfrak{A}}([c_0]_\sim, \dots, [c_{n-1}]_\sim) = [d]_\sim$$

if $T \vdash f(c_0, \dots, c_{n-1}) = d$ for $d \in \mathcal{C}_\tau$. For $R \in \mathcal{R}_\tau$ of arity n and $c_0, \dots, c_{n-1} \in \mathcal{C}_\tau$ then

$$([c_0]_\sim, \dots, [c_{n-1}]_\sim) \in R^{\mathfrak{A}} \qquad \text{iff} \qquad R(c_0, \dots, c_{n-1}) \in \mathrm{T}.$$

Of course we must check that f is actually a function, and that it is well-defined. Likewise we must also show that $R^{\mathfrak{A}}$ is well-defined.

To see that f is a function consider the formula $\varphi(x)$ given by $f(c_0, \dots, c_{n-1}) = x$. Since T has Henkin constants we have

$$\mathrm{T} \vdash \exists x \varphi(x) \longleftrightarrow \varphi(d)$$

for some $d \in \mathcal{C}_\tau$. Since T is complete and finitely satisfiable it must be the case that $\mathrm{T} \vdash \exists x f(c_0, \dots, c_{n-1}) = x$. Thus $\mathrm{T} \vdash \varphi(d)$ and so $f^{\mathfrak{A}}$ is defined. To show that $f^{\mathfrak{A}}$ and $R^{\mathfrak{A}}$ are *well*-defined uses the same style of arguments.

We now have a τ-structure $\mathfrak{A}$. Finally we show that $\mathfrak{A}$ is a model of T. We work by induction on the complexity of the sentence φ to show that $\mathfrak{A} \models \varphi$ if and only if $\varphi \in \mathrm{T}$. Without loss of generality we may assume φ is unnested.

If φ is an unnested atomic sentence then by construction of $\mathfrak{A}$ we see that $\mathfrak{A} \models \varphi$ if and only if $\varphi \in \mathrm{T}$.

If φ is $\theta \wedge \psi$, then $\mathfrak{A} \models \varphi$ iff $\mathfrak{A} \models \theta$ and $\mathfrak{A} \models \psi$ which by the inductive hypothesis happens iff $\theta \in \mathrm{T}$ and $\psi \in \mathrm{T}$. But $\theta \in \mathrm{T}$ and $\psi \in \mathrm{T}$ iff $\theta \wedge \psi \in \mathrm{T}$ since otherwise we violate the assumption that T is finitely satisfiable and complete. A similar argument works for the other boolean combinations.

Now suppose φ is $\exists x \theta$. Then

$$\begin{aligned} \mathfrak{A} \models \varphi \qquad & \text{iff} \qquad \exists a \in A,\ \mathfrak{A} \models \theta(a) && (1) \\ & \text{iff} \qquad \exists c \in \mathcal{C}_\tau,\ \mathfrak{A} \models \theta(c) && (2) \\ & \text{iff} \qquad \exists c \in \mathcal{C},\ \theta(c) \in \mathrm{T} && (3) \end{aligned}$$

So if $\mathfrak{A} \models \varphi$ then since T is complete and finitely satisfiable we must have $\varphi \in \mathrm{T}$. Furthermore if $\varphi \in \mathrm{T}$ then by the above biimplications and using that T has Henkin constants we see that $\mathfrak{A} \models \varphi$.

Thus, $\mathfrak{A}$ is a model of T. □

Exercise. Let $\tau = \{E\}$ where E is a binary relation symbol. Use the compactness theorem to show that $\mathfrak{A} \preccurlyeq \mathfrak{B}$ where $\mathfrak{A}$ and $\mathfrak{B}$ are τ-structures such that $E^{\mathfrak{A}}$ and $E^{\mathfrak{B}}$ are both equivalence relations. In $\mathfrak{A}$ there is exactly one equivalence class of size n for each $n \in \omega$, and $\mathfrak{B}$ extends $\mathfrak{A}$ by having one new infinite equivalence class.

Corollary (Upward Löwenheim-Skolem)**.** *If $\mathfrak{A}$ is infinite and λ any infinite cardinal and $|\mathfrak{A}| \leq \lambda$, then there exists $\mathfrak{B}$ such that $\mathfrak{A} \preccurlyeq \mathfrak{B}$ and $|\mathfrak{B}| = \lambda$.*

Proof. Let $\tau' \supseteq \tau$ be an expansion by constants

$$\mathcal{C}_{\tau'} := \mathcal{C}_\tau \cup \{c_\alpha : \alpha < \lambda\}.$$

Consider the theory

$$\mathrm{T} := \mathrm{Th}(\mathfrak{A}_A) \cup \{c_\alpha \neq c_\beta : \alpha < \beta\}.$$

We claim that T is finitely satisfiable: Let $S \subseteq \mathrm{T}$ be finite. Then S mentions only finitely many c_α's say, $\{c_{\alpha_1}, \dots, c_{\alpha_m}\}$. Let $\mathfrak{A}'$ be the τ'_A-structure with $\mathfrak{A}'|_{\tau_A} = \mathfrak{A}_A$ and $c_{\alpha_i}^{\mathfrak{A}'} := a_i$ where we pick distinct elements $a_1, \dots, a_m$ from A (which is possible since A is infinite). Then $\mathfrak{A}' \models S$.

By compactness there exists $\mathfrak{B}'$ a model of T. By construction $\mathfrak{B}'|_{\tau_A} \equiv \mathfrak{A}_A$ so $\mathfrak{A} \preccurlyeq \mathfrak{B}'|_\tau$ (by the elementary diagram lemma). Now $\{c_\alpha^{\mathfrak{B}} : \alpha < \lambda\} \subseteq B$ and so $|\mathfrak{B}| \geq \lambda$. If $|\mathfrak{B}|$ is too big we can use the Downward Löwenheim-Skolem theorem to get the right size. □