Math 225A – Model Theory

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General Information

These notes are based on a course in *Metamathematics* taught by Professor Thomas Scanlon at UC Berkeley in the Autumn of 2013. The course will focus on Model Theory and the course book is Hodges' *a shorter model theory*.

As with any such notes, these may contain errors and typos. I take full responsibility for such occurences. If you find any errors or typos (no matter how trivial!) please let me know at mps@berkeley.edu.

Lecture 8

Quantifier elimination continued

We will prove quantifier elimination for $\mathrm{Th}(\mathbb{N}, <)$. The proof will be very syntactic and quite heavy-handed. This should in part serve as motivation for the more structural methods of quantifier elimination that will be developed later in the course.

Theorem 1. *In $\mathscr{L}(<)$ the set Φ consisting of the atomic formulae together with*

$$\text{for every } n \in \mathbb{Z}_+ \qquad B_n(x, y) := \exists \bar{z}\ x < z_1 < \cdots < z_n < y$$

and

$$\text{for every } n \in \mathbb{Z}_+ \qquad L_n(x) := \exists \bar{z}\ x > z_1 > \cdots > z_n,$$

is an elimination set for $\mathrm{Th}(\mathbb{N}, <)$*. I.e. in the signature τ' where $\mathcal{C}_{\tau'} = \mathcal{F}_{\tau'} = \emptyset$ and*

$$\mathcal{R}_{\tau'} = \{<\} \cup \{B_n : n \in \mathbb{Z}_+\} \cup \{L_n : n \in \mathbb{Z}_+\}.$$

then the expansion $\mathrm{Th}_{\mathscr{L}(\tau')}(\mathbb{N}, <)$ has quantifier elimination.

Remark. $B_n(x, y)$ says 'there are n elements between x and y'. $L_n(x)$ says 'there are n elements less than x'.

Proof. By the lemma proved last time it suffices to show that if $\varphi(x, y_0, \ldots, y_{n-1}) \in \mathscr{L}(\tau')$ is quantifier-free with free variables amongst $x, \bar{y}$, then $\exists x \varphi(x, \bar{y})$ is equivalent to a quantifier-free formula in $\mathscr{L}(\tau')$.

Write φ in (the equivalent) disjunctive normal form as $\bigvee_i \bigwedge_j \theta_{i,j}$ where each $\theta_{i,j}$ is a literal. Now since the operator $\exists$ distributes over $\bigvee$ it suffices to eliminate quantifiers from the conjuncts. I.e. it suffices to show that $\exists x \bigwedge \theta_{i,j}$ is equivalent to some quantifier-free formula.

So from now on φ will be renamed to $\bigwedge \theta_{i,j}$.

Now we must figure out what the literals $\theta_{i,j}$ can possibly be. For example they can be of the following form, $L_n(y_i), L_n(x), \neg L_n(y_j), \neg L_n(x), B_n(x, y_i), \neg B_n(x, y_i)$, $x < y_i$, $\neg(x < y_i), \dots$.

We make another simplifying observation: φ is equivalent to a big disjunction over all possible "order relations" between the elements $y_0, \dots, y_{n-1}, x$ of the formula describing this given "order relation" conjoined with $\varphi(x, \bar{y})$. Here "order relation" means a possible way that the variables $y_0, \dots, y_{n-1}$ and x can be related via the $<$ relation symbol. For example one such order relation $\psi(x, \bar{y})$ could be

$$y_0 = y_1 = y_2 < y_3 < \cdots < y_i = \cdots = y_{n-1} < x.$$

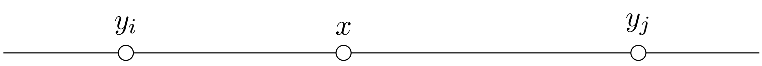
There are only finitely many different such order relations. Now the observation is that $\varphi(x, \bar{y})$ is equivalent to

$$\bigvee \{\psi(x, \bar{y}) \wedge \varphi(x, \bar{y}) : \psi(x, \bar{y}) \text{is an order relation}\}.$$

So we may assume that we have already pinned down completely the order relation of $y_0, \dots, y_{n-1}, x$. This means that any other order condition contained inside φ (for example one of the $\theta_{i,j}$ could say $x < y_i$ or $x = y_j$) will now either be redundant or explicitly contradictory with the order relation. If one of the literals $\theta_{i,j}$ is an explicit contradiction to the order relation then it is easy to find an equivalent quantifier-free formula, namely any false formula. If one of the literals $\theta_{i,j}$ is redundant then we need not worry about it.

Now we complete the proof by considering the remaining cases.

- If $\varphi \longrightarrow x{=}y_j$ for some $j \leq n-1$ then $\exists x\ \varphi(x, \bar{y})$ is equivalent to $\varphi(y_j, \bar{y})$, which is quantifier-free.
- So we may assume $\varphi \longrightarrow \bigwedge_i x \neq y_i$. Then there will be a single smallest interval where x is. I.e. $\varphi \longrightarrow y_i < x < y_j$ for some unique i, j such that $\varphi \longrightarrow \bigwedge_k \neg(y_i < y_k < y_j)$. Of course x could also be smaller than or greater than all the y_i's. This gives two more cases, but for convenience we shall allow "$y_i = \pm\infty$" and "$y_j = \pm\infty$". Pictorially we now fix ourselves in the following generic situation:



Now we must consider the other $\theta_{i,j}$'s in φ. I.e. the conditions which make use of the symbols L_n and B_n (with possible negations). Since we have fixed the order relation for φ we know that $\exists x \varphi$ if and only if there exists an x which satisfies each of the extra conditions individually. Thus it suffices to eliminate quantifiers from simple formulae of the form $\exists x \theta_{i,j}(x, \bar{y})$ for the cases where $\theta_{i,j}$ is one of the L_n or B_n (or negations thereof). Note that we need only concern ourselves with the $\theta_{i,j}$'s that have instances of x in them. There are four cases:

– $\exists x L_r(x)$ is equivalent to $L_{r+1}(y_i)$.
– $\exists x \neg L_r(x)$ is equivalent to $\neg L_{r-1}(y_j)$
– If $x < y_k$ then $\exists x B_t(x, y_k)$ is equivalent to $B_{t+1}(y_i, y_k)$. If $y_k < x$ then $B_t(x, y_k)$ is equivalent to $B_{t+1}(y_j, y_k)$.
– For $x < y_k$ then $\exists x \neg B_t(x, y_k)$ is equivalent to $\neg B_{t-1}(y_j, y_k)$. If $y_k < x$ then $\exists x \neg B_t(x, y_k)$ is equivalent to $\neg B_{t+1}(y_i, y_k)$.

This completes the proof. □

Once we have the compactness theorem one can use back-and-forth type arguments to greatly reduce the trouble with proving quantifier elimination results. The above proof demonstrates somewhat the idea behind proofs to come; we tried to "complete" the formula $\exists x \varphi(x, \bar{y})$ as much as possible so that there is only one formula to think about.

Skolem's theorem

We now move to more structural ideas. First we prove Skolem's theorem, showing that given any model we can find an elementary submodel of size $\leq$ the cardinality of the language.

Theorem 2. *For any τ-structure $\mathfrak{A}$ there exists $\mathfrak{B} \preccurlyeq \mathfrak{A}$ with $|\mathfrak{B}| \leq |\mathscr{L}(\tau)|$.*

Proof. Let $A = \mathrm{dom}(\mathfrak{A})$. We will build an increasing sequence $B_0 \subseteq B_1 \subseteq \cdots$ of subsets of A such that the union $B := \bigcup_i B_i$ will give us the domain of an elementary substructure of $\mathfrak{A}$.

At stage 0 set $B_0 := \emptyset$. At stage $n+1$ list all $\varphi(x)$ in $\mathscr{L}(\tau_{B_n,x})$ (i.e. φ has one free variable x and parameters from B_n). For each φ if $\mathfrak{A}_{B_n} \models \exists x \varphi(x)$ then let a_φ be a witness. Now set

$$B_{n+1} := B_n \cup \{a_\varphi : \varphi \in \mathscr{L}(\tau_{B_n,x}) \text{ and } \mathfrak{A}_{B_n} \models \exists x \varphi(x)\}$$

First let us check that $|B_n| \leq |\mathscr{L}(\tau)|$. For $n = 0$ this is clear. For $n+1$ by have by induction hypothesis that $|B_n| \leq |\mathscr{L}(\tau)|$. Then $|\mathscr{L}(\tau_{B_n})| = |\mathscr{L}(\tau)|$, which implies that

$$|B_{n+1}| \leq |B_n| + |\mathscr{L}(\tau)| \leq |\mathscr{L}(\tau)|.$$

So

$$|B| = |\bigcup_n B_n| \leq |\mathscr{L}(\tau)|.^1$$

[1]Here we are using the theorem from Set Theory that a union of an ω-chain of elements having cardinality $\leq \lambda$ (for $\lambda \geq \omega$) has cardinality $\leq \lambda$.

Finally we claim that B is the domain of an elementary substructure of $\mathfrak{A}$. It is the domain of a substructure because, by construction, it contains witnesses to statements of the form $\exists x\, x{=}c$ for each constant symbol c, and $\exists x f(b_1, \ldots, b_n){=}x$ for each function symbol f and $b_1, \ldots b_n \in B$. Finally it is an elementary substructure by the Tarski-Vaught Test. $\square$

As a simply corollary we have.

Corollary. *If $\mathfrak{A}$ is any τ-structure and λ a cardinal such that $|\mathscr{L}(\tau)| \leq \lambda \leq |\mathfrak{A}|$. Then there exists $\mathfrak{B} \preccurlyeq \mathfrak{A}$ such that $|\mathfrak{B}| = \lambda$.*

Proof. Let $Z \subseteq A$ be a subset with $|Z| = \lambda$. Consider the expansion τ_Z. Then by Skolem's theorem we find $\mathfrak{B}_Z \preccurlyeq \mathfrak{A}_Z$ with $|B_Z| \leq |\mathscr{L}(\tau_Z)|$. But we also have $|Z| \leq |B_Z|$ and so $B_Z = |Z| = \lambda$. Now let $\mathfrak{B} := \mathfrak{B}_Z|_\tau$ to get $\mathfrak{B} \preccurlyeq \mathfrak{A}$. $\square$

Skolem functions

Skolem himself proved his theorem by using what we shall call Skolemisation. The process will be used again and again.

Definition. A theory T in a signature τ has **Skolem functions** if for each formula $\varphi(\bar{x}, y) \in \mathscr{L}(\tau)$ there is a (not necessarily unique) function symbol f_φ with $\text{arity}(f_\varphi) = \text{length}(\bar{x})$ such that T contains the formula

$$\forall \bar{x}\, [\exists y\ \varphi(\bar{x}, y) \longleftrightarrow \varphi(\bar{x}, f_\varphi(\bar{x}))]\,.$$

So the function f_φ finds witnesses (depending on $\bar{x}$) whenever $\exists y \varphi(\bar{x}, y)$ is true.

Remark. Some writers replace function symbols by terms, so that "Skolem functions" are actually terms of the signature. In this way one can better handle the case where $\text{length}(\bar{x}) = 0$. We would need a "0-ary function symbol" which our definition does not allow. In our definition we can simply add dummy variables so that $\text{length}(\bar{x}) > 0$.

Remark. Some people say that T *has built in Skolem functions* if it has a definitional expansion with Skolem functions. Our notion of a theory with Skolem functions is more restrictive.

Example. If $\tau = \emptyset$ then the "theory of equality" is a theory *without* Skolem functions.

Example. The theory $\text{Th}(\mathbb{N}, +, \cdot, 0, 1, <)$ has built in Skolem functions (in the sense of the above remark) but does *not* have Skolem functions in our sense.

Next time we shall show how to add Skolem functions to our theories, a process called *Skolemisation.* A Corollary to this will be another proof of the Löwenheim-Skolem theorem.