Math 225A – Model Theory

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General Information

These notes are based on a course in *Metamathematics* taught by Professor Thomas Scanlon at UC Berkeley in the Autumn of 2013. The course will focus on Model Theory and the course book is Hodges' *a shorter model theory.*

As with any such notes, these may contain errors and typos. I take full responsibility for such occurences. If you find any errors or typos (no matter how trivial!) please let me know at mps@berkeley.edu.

Lecture 7

Definitional expansions continued

Let τ' be an extension of the signature τ. Let T be an τ-theory and T' be an τ'-theory, such that every model of T' has a reduct back to a model of T. Furthermore assume that for each new symbol in τ' we have a definition of that symbol in terms of $\mathscr{L}(\tau)$. I.e.

- For each $c \in \mathcal{C}_{\tau'} \setminus \mathcal{C}_\tau$ we have $\theta_c(x) \in \mathscr{L}(\tau_x)$,
- For each $f \in \mathcal{F}_{\tau'} \setminus \mathcal{F}_\tau$ we have $\gamma_f(\bar{x}, y)$,
- For each $R \in \mathcal{R}_{\tau'} \setminus \mathcal{R}_\tau$ we have $\psi_R(\bar{x})$,

and such that T says

- $\exists^{=1} x\ \theta_c(x)$ for each $c \in \mathcal{C}_{\tau'} \setminus \mathcal{C}_\tau$.
- $\forall \bar{x} \exists^{=1} y\ \gamma_f(\bar{x}, y)$ for each $f \in \mathcal{F}_{\tau'} \setminus \mathcal{F}_\tau$,

and such that T' says that these formulas formally define the constants, functions and relations, i.e.

- $\forall x \theta_c(x) \leftrightarrow x = c$ for each $c \in \mathcal{C}_{\tau'} \setminus \mathcal{C}_\tau$
- $\forall \bar{x} \forall y\ [\gamma_f(\bar{x}, y) \leftrightarrow f(\bar{x}) = y]$ for each $f \in \mathcal{F}_{\tau'} \setminus \mathcal{F}_\tau$,
- $\forall \bar{x}\ [\psi_R(\bar{x}) \leftrightarrow R(\bar{x})]$ for each $R \in \mathcal{R}_{\tau'} \setminus \mathcal{R}_\tau$.

Given $\tau \subseteq \tau'$, T and T' as above, then we have the restriction map

$$\mathrm{Res}_\tau : \mathrm{Str}(\tau') \longrightarrow \mathrm{Str}(\tau).$$

Now by assumption we can restrict the restriction map to $\mathrm{Mod}(T')$. The induced map on $\mathrm{Mod}(T')$ has range inside $\mathrm{Mod}(T)$ by assumption. With this setup we state the following proposition.

$$\begin{array}{ccc}
\mathrm{Str}(\tau') & \xrightarrow{\mathrm{Res}_\tau} & \mathrm{Str}(\tau) \\
\Big\uparrow \iota & & \Big\uparrow \iota \\
\mathrm{Mod}(T') & \xrightarrow[\mathrm{Res}_\tau]{} & \mathrm{Mod}(T)
\end{array}$$

Proposition. *Given $\tau \subseteq \tau'$, T and T' as above, the induced map* $\mathrm{Mod}(T') \longrightarrow \mathrm{Mod}(T)$ *is a bijection (of classes).*

Proof. Suppose $\mathfrak{A} \models T'$. We check that $\mathfrak{A}|_T \models T$. Let $\varphi \in T$. There are two cases.

- φ is $\exists^{=1}x\ \theta_c(x)$ for some constant c. For any $a \in \mathrm{dom}(\mathfrak{A})$ then $\mathfrak{A}_a \models \theta_c(a)$ if and only if $a = c^{\mathfrak{A}}$ thus $\mathfrak{A} \models \exists^{=1}x\ \theta_c(x)$ so $\mathfrak{A}|_\tau \models \varphi$ as well.
- φ is $\forall\bar{x}\exists^{=1}y\ \gamma_f(\bar{x}, y)$ for some function symbol f. Now $\mathfrak{A} \models \forall\bar{x}\exists^{=1}y\ f(\bar{x}) = y$ so

$$\mathfrak{A} \models \forall\bar{x}\forall y\ [f(\bar{x}) = y \leftrightarrow \gamma_f(\bar{x}, y)]$$

  so $\mathfrak{A} \models \forall\bar{x}\exists^{=1}y\ \gamma_f(\bar{x}, y)$ which implies that $\mathfrak{A}|_\tau \models \varphi$.

Thus $\mathfrak{A}|_\tau \models T$.
Conversely, suppose $\mathfrak{A} \models T$. We want to expand $\mathfrak{A}$ to some τ'-structure $\mathfrak{A}'$ which is a model of T'.

- For $c \in \mathcal{C}'_\tau \setminus \mathcal{C}_\tau$ define $c^{\mathfrak{A}'}$ to be the unique $a \in \mathrm{dom}(\mathfrak{A})$ such that $\mathfrak{A}_a \models \theta_c(a)$.
- For $f \in \mathcal{F}'_\tau \setminus \mathcal{F}_\tau$ then we define $f^{\mathfrak{A}'}$ by

$$f^{\mathfrak{A}'}(\bar{a}) = b \Leftrightarrow \mathfrak{A} \models \gamma_f(\bar{a}, b).$$

  Note that this actually defines a function because of what γ_f says.
- For $R \in \mathcal{R}'_\tau \setminus \mathcal{R}_\tau$ we let

$$R^{\mathfrak{A}'} := \{\bar{a} \in \mathrm{dom}(\mathfrak{A})^{\mathrm{arity}(R)} : \mathfrak{A}_{\bar{a}} \models \psi_R(\bar{a})\}.$$

This makes $\mathfrak{A}'$ into an τ'-structure which is a model of T' and $\mathfrak{A}'|_\tau = \mathfrak{A}$. This completes the proof. $\square$

Atomisation/Morleyisation

The following construction is usually called *Morleyisation.* Hodges however, calls it *Atomisation.* He points out that Thoralf Skolem used this construction before Morley did. Since the term "Skolemisation" has a different meaning, Hodges decides that "atomisation" is both more correct and more descriptive.

Given a signature τ we build a new signature τ' (which will *not* be an expansion). Let $\mathcal{C}_{\tau'} = \mathcal{F}_{\tau'} = \emptyset$ and

$$\mathcal{R}_{\tau'} := \{R_{(\varphi,n)} : \varphi \in \mathscr{L}(\tau_{\{x_i : i<n\}})\}$$

with $\text{arity}(R_{(\varphi,n)}) = n$[1].

We now make a definitional expansion from a theory in τ to a theory in $\tau' \cup \tau$. Consider $T = \emptyset$ the empty theory in $\mathscr{L}(\tau)$, and T' a theory in $\mathscr{L}(\tau \cup \tau')$ given by

$$T' := \{\forall \bar{x}[R_{(\varphi,n)}(\bar{x}) \leftrightarrow \varphi(\bar{x})] : \varphi \in \mathscr{L}(\tau_{\{x_i : i<n\}})\}.$$

Then T and T' trivially satisfy the conditions for the definitional expansions as in the above section, since there are no new constant symbols and no new function symbols. By the proposition we proved for definitional expansions, each τ-structure $\mathfrak{A}$ (i.e. and model of $T = \emptyset$) admits a unique definitional expansion to a $\tau \cup \tau'$-structure $\mathfrak{A}'$ such that $\mathfrak{A}' \models T'$.

Definition. With the setup as describe above, the **atomisation** of $\mathfrak{A}$ is the reduct of $\mathfrak{A}'$ down to τ', i.e. $\mathfrak{A}^{Atom} := \mathfrak{A}'|_{\tau'}$.

Proposition. *Let $\mathfrak{A}^{Atom}$ be the atomisation of $\mathfrak{A}$. Then every definable set in $\mathfrak{A}^{Atom}$ is defined by an atomic τ'-formula*

Proof. This is true by definition of definitional expansions. Since any subset $X \subseteq \text{dom}(\mathfrak{A}^{Atom})^n = \text{dom}(\mathfrak{A})^n$ is $\mathscr{L}(\tau')$-definable if and only if it is $\mathscr{L}(\tau)$-definable. □

Remark. Depending on the definition one has of atomic formula we may need to assume that the definable sets in the propostion are defined in at least one variable. This is a necessary assumption if one does not count true ($\top$) and false ($\bot$) as atomic sentences.

Corollary. *If $\mathfrak{A}' \subseteq \mathfrak{B}'$ and $\mathfrak{A}', \mathfrak{B}' \models T'$ then $\mathfrak{A}' \preccurlyeq \mathfrak{B}'$ and $\mathfrak{A} \preccurlyeq \mathfrak{B}$ where $\mathfrak{A} := \mathfrak{A}'|_{\tau}$ and $\mathfrak{B} := \mathfrak{B}|_{\tau}$.*

Proof. We use the Tarski-Vaught Criterion, namely that $\mathfrak{A}' \preccurlyeq \mathfrak{B}'$ if and only if, for any formula $\theta(x) \in \mathscr{L}(\tau_{A'})$ $\mathfrak{A}' \models \exists x\theta(x)$ iff $\mathfrak{B}' \models \exists x\theta$. The forward direction is immediate. The backwards direction we may prove by induction on the complexity of our formula. Suppose φ is the τ-formula. We must show that $\mathfrak{A}_{\bar{a}} \models \varphi(\bar{a})$ iff $\mathfrak{B}_{\bar{a}} \models \varphi(\bar{a})$.

- For φ is atomic $\mathfrak{A} \subseteq \mathfrak{B}$
- For $\varphi \equiv \neg\psi$ note that negation preserves the biimplication.

[1]Many authors do not include the subscript n. They simply write R_{φ} without specifying the arity.

- The conjunction and disjunction are immidiate.
- For $\varphi \equiv \exists x \psi$ this is exactly our hypothesis.

$\square$

Remark. The above corollary only holds if we allow $\top$ and $\bot$ as atomic formulas. Otherwise we must assume that $\mathfrak{A} \equiv_{\mathscr{L}(\tau)} \mathfrak{B}$.

The atomisation process is useful for simplifying some arguments. Furthermore if one is only interested in the class of definable sets of a given structure then the atomisation is also useful since it has the same class, only this time each set is defined by atomic formulas. But if one actually wants to determine what these definable sets are, then the atomisation is completely useless.

We now give some examples and non-examples of elementary substructures.

Example. Let $\tau = \{<\}$. Let $\mathfrak{B} = (\mathbb{Q}, <)$ and $\mathfrak{A} = (\mathbb{Z}[\frac{1}{2}], <)$. Then $\mathfrak{A} \preccurlyeq \mathfrak{B}$. We will not prove this now.

Example. Let $\tau = \{<\}$. Let $2\mathbb{Z} = (2\mathbb{Z}, <)$ and $\mathbb{Z} = (\mathbb{Z}, <)$. Then $2\mathbb{Z} \subseteq \mathbb{Z}$ as τ-structures. Furthermore $2\mathbb{Z} \models \forall x \neg(x < 4 \wedge 2 < x)$ and $\mathbb{Z} \models \exists x(x < 4 \wedge 2 < x)$ so $2\mathbb{Z} \not\preccurlyeq \mathbb{Z}$. However, since $2\mathbb{Z}$ and $\mathbb{Z}$ are isomorphic as $\mathscr{L}(\tau)$-structures we do have that $2\mathbb{Z} \equiv_{\mathscr{L}(\tau)} \mathbb{Z}$.

Question. Does there exists $\mathfrak{B}$ a τ-structure and $\mathfrak{A} \subseteq \mathfrak{B}$ and $f : \mathfrak{B} \longrightarrow \mathfrak{A}$ a definable isomorphism (i.e. the graph of f is a definable set) such that $\mathfrak{A} \not\preccurlyeq \mathfrak{B}$? [Hint: $S : \omega \longrightarrow \mathbb{Z}_+$ the succesor map.]

Question. Is it true that given $\mathfrak{B}$ and $\mathfrak{A} \subseteq \mathfrak{B}$ and a definable isomorphism $f : \mathfrak{B} \longrightarrow \mathfrak{A}$ such that $\mathfrak{A} \preccurlyeq \mathfrak{B}$ then we must have $\mathfrak{A} = \mathfrak{B}$?

In some sense the notion of an extension of a structure is the right notion from the level of the atomic formulas. But if one is interested in the definable sets and how to interpret the formulae from one structure to another, then *elementary* extension is the right notion.

Quantifier Elimination

In practice one does not use the atomisation procedure to gain information about the definable sets. To actually gain information one can hope to find a reasonable class of formulae from which every definable set can be defined.

Definition. Given a signature τ and a class $\mathcal{K}$ of τ-structures. A set $\Phi \subseteq \mathscr{L}(\tau)$ of τ-formulae is an **elimination set for $\mathcal{K}$** if

- for every formula $\psi(\bar{x}) \in \mathscr{L}(\tau)$ (with at least one free variable) there exists a boolean combination $\varphi(\bar{x})$ of formulae from Φ such that for all $\mathfrak{A} \in \mathcal{K}$ we have $\mathfrak{A} \models \forall\bar{x}(\varphi \leftrightarrow \psi)$.

We say that $\mathcal{K}$ has **quantifier elimination** if we can take the elimination set Φ to be the collection of atomic formulae.

Remark. For any τ and any $\mathcal{K}$ there always exists an elimination set, namely $\mathscr{L}(\tau)$ itself.

Remark. Relativising the atomisation construction to $\Phi \subseteq \mathscr{L}(\tau)$, then Φ is an elimination set for $\mathcal{K}$ if and only if each definable expansion of structures of $\mathcal{K}$ to the relative atomisation has quantifier elimination.

Definition. For a structure $\mathfrak{A}$, we say that $\mathfrak{A}$ **eliminates quantifiers** if $\mathcal{K}$ does so. Similarly if $\mathcal{K} = \mathrm{Mod}(T)$ then we say T **eliminates quantifiers**.