Math 225A – Model Theory

Speirs, Martin

Autumn 2013

General Information

These notes are based on a course in *Metamathematics* taught by Professor Thomas Scanlon at UC Berkeley in the Autumn of 2013. The course will focus on Model Theory and the course book is Hodges' *a shorter model theory.*

As with any such notes, these may contain errors and typos. I take full responsibility for such occurences. If you find any errors or typos (no matter how trivial!) please let me know at mps@berkeley.edu.

Lecture 15

Elimination of Imaginaries

Definition. An **imaginary element** of a τ-structure $\mathfrak{A}$ is a class $[a]_E$ where $a \in A^n$ and E is a definable equivalence relation on A^n.

So an imaginary element can be thought of as an element of a quotient of a definable set by a definable equivalence relation. Thus thinking in terms of the category of definable sets[1], passing to the imaginaries means allowing this category to have quotients.

Example (Trivial equivalence relation). If $a \in A^n$ then we may identify a with the class $[a]_{=_{A^n}}$ under the (definable) equivalence relation given simply by equality.

Definition. A τ-structure $\mathfrak{A}$ **eliminates imaginaries** if, for every definable equivalence relation E on A^n there exists definable function $f : A^n \longrightarrow A^m$ such that for $x, y \in A^n$ we have

$$xEy \iff f(x) = f(y)$$

Remark. The definition given above is what Hodges calls *uniform elimination of imaginaries.*

Remark. If $\mathfrak{A}$ eliminates imaginaries, then for any definable set X and definable equivalence relation E on X, there is a definable set Y and a definable bijection $f : X/E \longrightarrow Y$. Of course this is not literally true, we should rather say that there is a definable map $f' : X \longrightarrow Y$ such that f' is invariant on the equivalence classes defined by E.

So elimination of imaginaries is saying that quotients exists in the category of definable sets.

[1]This is the category whose objects are definable sets and morphism are definable functions.

Remark. If $\mathfrak{A}$ eliminates imaginaries then for any imaginary element $[a]_E = \tilde{a}$ there is some tuple $\hat{a} \in A^m$ such that $\tilde{a}$ and $\hat{a}$ are **interdefinable** i.e. there is a formula $\varphi(x, y)$ such that

- $\mathfrak{A} \models \varphi(a, \hat{a})$,
- If $a' E a$ then $\mathfrak{A} \models \varphi(a', \hat{a})$,
- If $\varphi(b, \hat{a})$ then bEa,
- If $\varphi(a, c)$ then $c = \hat{a}$.

To get the formula φ we use the function f given by the definition of elimination of imaginaries, and let $\varphi(x, y)$ be $f(x) = y$, (note: then $\hat{a} = f(a)$).

Almost conversely, if for every $\mathfrak{A}' \equiv \mathfrak{A}$ every imaginary in $\mathfrak{A}'$ is interdefinable with a **real** (i.e. non-imaginary) tuple then $\mathfrak{A}$ eliminates imaginaries. We will prove this after proving the compactness theorem.

Example. For any structure $\mathfrak{A}$, every imaginary in $\mathfrak{A}_A$ is interdefinable with a sequence of real elements.

Example. Let $\mathfrak{A} = (\mathbb{N}, <, \equiv \pmod 2)$. Then $\mathfrak{A}$ eliminates imaginaries. For example, to eliminate the "odd/even" equivalence relation, E, we can define $f : \mathbb{N} \longrightarrow \mathbb{N}$ by mapping x to the least z such that xEz. I.e. f is define by the formula

$$f(x) = y \iff xEy \wedge \forall z[xEz \longrightarrow y < z \vee y = z].$$

In the above example we claim furthermore $\mathfrak{A}$ eliminates all other equivalence relations. This is because $\mathfrak{A}$ has *definable choice functions.*

Definition. $\mathfrak{A}$ has **definable choice functions** if for any formula $\theta(\bar{x}, \bar{y})$ there is a definable function $f(\bar{y})$ such that

$$\forall \bar{y} \exists \bar{x} [\theta(\bar{x}, \bar{y}) \longleftrightarrow \theta(f(\bar{y}), \bar{y})]$$

(i.e. f is a *skolem function* for θ) and such that

$$\forall \bar{y} \forall \bar{z} [\forall \bar{x} (\theta(\bar{x}, \bar{y}) \longleftrightarrow \theta(\bar{x}, \bar{z})) \longrightarrow f(\bar{y}) = f(\bar{z})]$$

Proposition. *If $\mathfrak{A}$ has definable choice functions then $\mathfrak{A}$ eliminates imaginaries.*

Proof. Given a definable equivalence relation E on A^n let f be a definable choice function for $E(x, y)$. Since E is an equivalence relation we have $\forall y E(f(y), y)$ and

$$\forall y, z \; ([y]_E = [z]_E \longrightarrow f(y) = f(z))$$

thus $f(y) = f(z) \iff yEz$. □

Example (continued). We now see that $\mathfrak{A} = (\mathbb{N}, <, \equiv \pmod 2)$ eliminates imaginaries. Basically since $\mathfrak{A}$ is well ordered, we can find a least element to witness membership of definable sets, hence we have definable choice functions.

Question. Suppose $\mathfrak{A}$ has Skolem functions. Must $\mathfrak{A}$ eliminate imaginaries?

Example. $\mathfrak{A} = (\mathbb{N}, \equiv \pmod 2)$ does *not* eliminate imaginaries.

First note that the only definable subsets of $\mathbb{N}$ are $\emptyset, \mathbb{N}, 2\mathbb{N}$ and $(2n+1)\mathbb{N}$. This is because $\mathfrak{A}$ has an automorphisms which switches $(2n+1)\mathbb{N}$ and $2\mathbb{N}$.

Now suppose $f : \mathbb{N} \longrightarrow \mathbb{N}^M$ eliminates the equivalence relation $\equiv \pmod 2$, i.e.

$$f(x) = f(y) \iff y \equiv x \pmod 2.$$

Then range(f) is definable and has cardinality 2. Since there are no definable subsets of $\mathbb{N}$ of cardinality 2, we must have $M > 1$. Now let $\pi : \mathbb{N}^M \longrightarrow \mathbb{N}$ be a projection. Then $\pi(\text{range}(f))$ is a finite nonempty definable subset of $\mathbb{N}$. But no such set exists.

Note that if we allow parameters in defining subsets, then $\mathfrak{A}$ does eliminate imaginaries.

Example. Consider a vector space V over a field K. We will put these together into a two-sorted structure $(V, K, +_V, 0_V, +_K, \cdot_K, \cdot_{K,V}, 0_K, 1_K)$ here the functions and constant are suitably defined. Now define, for $v, w \in V$,

$$v \sim w \iff \exists \lambda \in K \setminus 0 \ \lambda v = w$$

Then $V/\sim$ is the projective space $\mathbb{P}(V)$.

Question. Can we eliminate imaginaries in this case?

Proposition. *If the τ-structure $\mathfrak{A}$ eliminates imaginaries, then $\mathfrak{A}_A$ eliminates imaginaries.*

Proof. The idea is that an equivalence relation with parameters can be obtained as a fiber of an equivalence relation in more variables. More precisely, let $E \subseteq A^n$ be an equivalence relation definable in $\mathfrak{A}_A$. Let $\varphi(x, y; z) \in \mathscr{L}(\tau)$ and $a \in A^l$ be such that

$$xEy \iff \mathfrak{A} \models \varphi(x, y; a).$$

Now define

$$\psi(x, u, y, v) = \begin{cases} u = v \wedge \text{“}\varphi \text{ defines an equivalence relation“} & \text{or} \\ u \neq v & \text{or} \\ \text{“}\varphi(x, y, v) \text{ does not define an equivalence relation“} & \end{cases}$$

Where “φ defines an equivalence relation“ is clearly first-order expressible. Now ψ defines an equivalence relation on A^{n+l}. Letting $f : A^{n+l} \longrightarrow A^M$ eliminate ψ, then $f(-, a)$ eliminates E. □

Multi-Sorted Structures, $\mathfrak{A}^{eq}$

We saw that atomisation was a way to force elimination of quantifiers. Similarly one can force elimination of imaginaries, provided one is willing to work in a multi-sorted logic.

Given a τ-structure $\mathfrak{A}$ we will construct $\mathfrak{A}^{eq}$ as follows. For each definable equivalence relation E on A^n we have a sort S_E and a function symbol π_E interpreted as

$$\pi_E^{\mathfrak{A}^{eq}} : A^n \longrightarrow S_E^{\mathfrak{A}^{eq}} := A^n/E$$

mapping a to $[a]_E$. This shows $\mathfrak{A}^{eq}$ is interpreted[2] in $\mathfrak{A}$.

Conversely, $\mathfrak{A}$ can be interpreted in $\mathfrak{A}^{eq}$. Let $\partial_{()=}(x)$ be $x \in S_{=_A}$. Given an unnested τ-formula $\varphi(x_0, \ldots, x_{l-1})$ consider E_φ defined by

$$\bar{x} E_\varphi \bar{y} \quad \Longleftrightarrow \quad (\varphi(\bar{x}) \longleftrightarrow \varphi(\bar{y}))$$

then we have

$$\pi_{E_\varphi} : A^l \longrightarrow S_{E_\varphi}$$

This almost works.

Question. How can we define, in $\mathfrak{A}^{eq}$, the class $[\bar{a}]_{E_\varphi}$ where $\mathfrak{A} \models \varphi(\bar{a})$?

[2]Here we mean interpreted in the sense of multi sorted structures.