Math 225A – Model Theory

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Autumn 2013

General Information

These notes are based on a course in *Metamathematics* taught by Professor Thomas Scanlon at UC Berkeley in the Autumn of 2013. The course will focus on Model Theory and the course book is Hodges' *a shorter model theory.*

As with any such notes, these may contain errors and typos. I take full responsibility for such occurences. If you find any errors or typos (no matter how trivial!) please let me know at mps@berkeley.edu.

Lecture 6

We discuss the basic idea of comparing different structures and ways of regarding the same structure in different languages. At one level this allows us to completely forget about syntax and focus on the definable sets. On another level it brings the syntax back to the fore because we will have specific ways of referring, to specific sets which might appear, as though they are actually part of the language. Hodges calls this *atomisation* although most people call it *Morleyisation.* As Hodges points out Skolem introduced the method before Morley.

Last Time

Let us first recall briefly the chain construction. We have a chain $(\mathfrak{A}_i)_{i\in I}$ of τ-structures indexed by a totally ordered set $(I,<)$. This is a functor from the category $(I,<)$ to the category $\mathrm{Str}(\tau)$ of τ-structures. The content of this rephrasing is just that and arrow $i<j$ is mapped to an arrow $\mathfrak{A}_i \subseteq \mathfrak{A}_j$. Given this chain we may form the union $\bigcup \mathfrak{A}_i$ which as its domain is the union of the domains of $\mathfrak{A}_i$ and which is given the natural τ-structure.

We proved last time that if φ is an $\forall_2$ sentence in $\mathscr{L}(\tau)$ and if for all i we have $\mathfrak{A}_i \models \varphi$ then $\bigcup \mathfrak{A}_i \models \varphi$. This proposition is a slight elaboration on the proposition that $\exists_1$ sentence "go up".

Theories and Models

Definition. If $\mathcal{K}$ is a class of τ-structures, then $\mathrm{Th}(\mathcal{K})$ is the set of all τ sentences φ such that for all $\mathfrak{A} \in \mathcal{K}$ we have $\mathfrak{A} \models \varphi$. I.e.

$$\mathrm{Th}(\mathcal{K}) := \{\varphi \in \mathscr{L}(\tau) : \varphi \text{ is a sentence } \forall \mathfrak{A} \in \mathcal{K}\ \mathfrak{A} \models \varphi\}$$

Definition. If T is a set of τ-sentences, then $\mathrm{Mod}(T)$ is the class of τ-structures $\mathfrak{A}$ such that $\mathfrak{A} \models T$. I.e.

$$\mathrm{Mod}(T) := \{\mathfrak{A} \in \mathrm{Str}(\tau) : \mathfrak{A} \models T\}$$

$$
\begin{tikzcd}
\text{classes of } \tau\text{-structures} \arrow[rr, bend left, "\mathrm{Th}(-)"] & & \text{sets of } \tau\text{-sentences} \arrow[ll, bend left, "\mathrm{Mod}(-)"]
\end{tikzcd}
$$

One immediately asks whether $\mathrm{Th}(-)$ and $\mathrm{Mod}(-)$ are each others inverses? They are not. But they are connected[1]. Indeed we have, by definition, that

$$\mathrm{Th}(\mathrm{Mod}(T)) \supseteq T \tag{1}$$

and

$$\mathrm{Mod}(\mathrm{Th}(\mathcal{K})) \supseteq \mathcal{K}. \tag{2}$$

Both inclusions may be strict.

Definition. Given a theory T and a sentence φ we say that T **semantically implies** φ, written $T \models \varphi$ iff any model of T is also a model of φ.

Given a theory T the set $\mathrm{Th}(\mathrm{Mod}(T))$ is the set of **semantic consequences of** T. It is the set of sentences that are satisfied by all models of T, i.e.

$$\mathrm{Th}(\mathrm{Mod}(T)) = \{\varphi : T \models \varphi\}.$$

Similarly $\mathrm{Mod}(\mathrm{Th}(\mathcal{K}))$ is the smallest definable class of structures containing $\mathcal{K}$.

Notation. If $\mathcal{K}$ is the singleton class $\{\mathfrak{A}\}$, then we write $\mathrm{Th}(\mathfrak{A})$ instead of $\mathrm{Th}(\{A\})$.

Definition. We say two structures $\mathfrak{A}$ and $\mathfrak{B}$ are **elementarily equivalent**, written $\mathfrak{A} \equiv \mathfrak{B}$, if $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})$.

Definition. If $\mathfrak{A} \subseteq \mathfrak{B}$ then $\mathfrak{A}$ is an **elementary substructure** of $\mathfrak{B}$, written $\mathfrak{A} \preccurlyeq \mathfrak{B}$ if the inclusion map preserves all formulas of $\mathscr{L}(\tau)$. Such an inclusion map is called an **elementary inclusion**.

See lecture 7 for an example where $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A} \equiv \mathfrak{B}$ but $\mathfrak{A} \not\preccurlyeq \mathfrak{B}$.

Definition. We say two theories S and T are **equivalent** if $\mathrm{Mod}(T) = \mathrm{Mod}(S)$.

[1] They form a Galois connection between the "posets" $(\mathcal{P}(\mathrm{Str}(\tau)), \supseteq)$ and $(\mathcal{P}(\mathscr{L}(\tau)), \subseteq)$, except that $\mathrm{Str}(\tau)$ is not a set and so neither is $\mathcal{P}(\mathrm{Str}(\tau))$!

We may often find that $\mathrm{Mod}(\mathrm{Th}(\mathcal{K}))$ strictly contains $\mathcal{K}$. For instance if $\mathcal{K}$ is a singleton $\{\mathfrak{A}\}$ where $\mathfrak{A}$ is some τ-structure, then $\mathrm{Mod}(\mathrm{Th}(\mathcal{K}))$ will contain all those τ-structures which are elementarily equivalent to $\mathfrak{A}$. So if there exist models, $\mathfrak{B}$, that are elementarily equivalent to $\mathfrak{A}$, i.e. $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})$ but such that $\mathfrak{A} \not\cong \mathfrak{B}$ then $\mathrm{Mod}(\mathrm{Th}(\mathfrak{A}))$ will properly contain $\{\mathfrak{A}\}$. In fact, given τ, there *always* exist $\mathfrak{A}$ and $\mathfrak{B}$ that are not isomorphic yet elementarily equivalent.

Proposition. *There exist $\mathfrak{A}$ and $\mathfrak{B}$ two τ-structures such that $\mathfrak{A} \not\cong \mathfrak{B}$ but $\mathfrak{A} \equiv \mathfrak{B}$.*

Proof. Consider the restricted functor $\mathrm{Th} : \mathrm{Str}(\tau) \longrightarrow \mathcal{P}(\mathscr{L}(\tau))$. Now $\mathrm{Str}(\tau)$ is a class (there are as many τ-structures as there are sets) and $\mathcal{P}(\mathscr{L}(\tau))$ is a set of cardinality at most $2^{|\mathscr{L}(\tau)|}$. By the Pigeon-hole-principle this is not injective. Even considering $\mathrm{Str}(\tau)$ up to isomorphism it is still a class since isomorphism preserves cardinality. Thus there are $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \not\cong \mathfrak{B}$ and yet $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})$. $\square$

We shall prove much stronger results than this later in the course.

The kinds of classes of structures that we will be most interested in will be those that appear as the classes of models of some theory T.

Definition. A class of τ-structures, $\mathcal{K}$, is an **elementary class** if $\mathcal{K} = \mathrm{Mod}(T)$ for some T. In this case we say that T **axiomatizes** $\mathcal{K}$.

Definition. Let $\mathcal{K}$ be a class of τ-structures. Then $\mathfrak{A} \in \mathcal{K}$ is **existentially closed in** $\mathcal{K}$ (or "e.c. in $\mathcal{K}$") if; given any $\mathfrak{B} \in \mathcal{K}$ with $\mathfrak{A} \subseteq \mathfrak{B}$ then, for every $\exists_1$ sentence ψ in $\mathscr{L}(\tau_A)$, if $\mathfrak{B} \models \psi$ then $\mathfrak{A} \models \psi$.

So a structure is existentially closed if you have already put in all the witnesses.

Theorem 1. *If $\mathcal{K} = \mathrm{Mod}(T)$ where T is $\forall_2$-axiomatizable, then for all $\mathfrak{A} \in \mathcal{K}$ there exists some $\mathfrak{B} \in \mathcal{K}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{B}$ is existentially closed.*

Proof. Given $\mathfrak{A} \in \mathcal{K}$ we build a chain of models $(\mathfrak{A}_n)_{n\in\omega}$ in $\mathcal{K}$ and then take the union. Let $\mathfrak{A}_0 := \mathfrak{A}$. We construct $\mathfrak{A}_1$ as follows. Let $\{\varphi_i \in \mathscr{L}(\tau_{A_0}) : \varphi_i \text{ is } \exists_1\text{-sentence}\}$ be an enumeration of the existential sentences with parameters from $A_0 = A$. Now let $\mathfrak{A}_{1,0} := \mathfrak{A}_1$ and for φ_0 we ask whether there exists any $\mathfrak{B} \in \mathcal{K}$ with $\mathfrak{B} \models \varphi_0(\bar{a})$ and $\mathfrak{A}_{1,0} \not\models \varphi_0$, if so then let $\mathfrak{A}_{1,1} := \mathfrak{B}$. Now at stage i we ask the same question for φ_i and if $\mathfrak{A}_{1,i}$ is not existentially closed with respect to φ_i then pick some $\mathfrak{B} \in \mathcal{K}$ that witnesses this and let $\mathfrak{A}_{1,i+1} := \mathfrak{B}$. Thus we get a chain of order type equal to the order type of $\mathscr{L}(\tau_{A_0})$. We take the union of this chain. This union is $\mathfrak{A}_1$. Now $\mathfrak{A}_1 \in \mathcal{K}$ since $\forall_2$ sentences are preserved in unions of chains. Likewise at stage n we construct $\mathfrak{A}_{n+1}$ by going through all sentences $\varphi_i(\bar{a})$ with parameters from A_n. At each stage we have $\mathfrak{A}_n \in \mathcal{K}$ and $\mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}$ by construction. Now we take the union of the ω-chain $(\mathfrak{A}_n)_{n\in\omega}$

$$\mathfrak{B} := \bigcup_{n\in\omega} \mathfrak{A}_n$$

Then $\mathfrak{B}$ is again in $\mathcal{K}$ by the preservation of $\forall_2$-sentences in chains. Also $\mathfrak{B}$ is existentially closed since given any $\exists_1$ sentence φ with parameters from B then since there are only finitely many of these parameters occurring in φ we have that $\varphi = \varphi_i$ for some $\varphi_i \in \mathscr{L}(\mathfrak{A}_n)$ for some n. At stage n we ensured that $\mathfrak{A}_{n+1} \subseteq \mathfrak{B}$ is existentially closed with respect to φ. This finishes the proof. $\square$

Example (Linear orders). Let τ be the signature $\mathcal{C}_\tau = \mathcal{F}_\tau = \emptyset$ and $\mathcal{R}_\tau = \{<\}$. Let T be the theory of linear orders.

- Let $\mathfrak{A} = (\omega, <)$ is *not* existentially closed. To see this let ψ be $\exists x(0 < x < 1)$ then take the natural extension of $\mathfrak{A}$ by adding $\frac{1}{2}$ to the set. Call this τ-structure $\mathfrak{B}$ then $\mathfrak{B} \models \psi$, and $\mathfrak{A} \subseteq \mathfrak{B}$ but $\mathfrak{A} \not\models \psi$.
- Let $\tilde{\mathfrak{A}}$ have domain $\{\frac{a}{2^n} : a \in \mathbb{N}, n \in \mathbb{N}\}$, with the natural order. Then $\tilde{\mathfrak{A}}$ is *not* existentially closed. For instance $(\mathbb{R}, <) \models \exists x\ x < 0$ and $\tilde{\mathfrak{A}} \not\models \exists x\ x < 0$.
- $(\mathbb{R}, <)$ *is* existentially closed. This requires a bit of work to show.

Example (Fields). Let τ be the signature of fields. An existentially closed field is E.C. if and only if it is algebraically closed

Example (Groups). It is difficult to describe explicitly the E.C. groups. Of course one can give examples of equations that are necessarily true in E.C. groups, for instance $\forall x \exists y\ y^n = x$.

In fact the class of E.C. groups cannot be axiomatized. We can however axiomatize the class of E.C. fields and the class of E.C. linear orders.

Unnested formulae

Definition. An **unnested atomic formula** is one of the form

- $x = c$, for $c \in \mathcal{C}_\tau$ and x a variable.
- $F\bar{x} = y$ where $F \in \mathcal{F}_\tau$ and $\bar{x}, y$ are variables.
- $R\bar{x}$, where $R \in \mathcal{R}_\tau$ and $\bar{x}$ are variables.
- $x = y$, where x and y are variables.

An **unnested formula** is built from the unnested atomic formulae by the usual rules.

Lemma. *Every formula $\varphi \in \mathscr{L}(\tau)$ is equivalent to some unnested formula $\tilde{\varphi}$. In fact if φ is atomic then $\tilde{\varphi}$ may be take to be either $\exists_1$ or $\forall_1$, and if φ is $\forall_n$ or $\exists_n$ then $\tilde{\varphi}$ may be taken to have the same quantifier complexity.*

Proof. Whenever some term contains a function symbol applied to something unnested we will strip of the function symbol and replace it by a new variable.

Let us start with φ an atomic formula. We will show that there is an equivalent existential formula where each of the sub-formulae have terms that are no more complicated than the one we had before (and at least one has complexity strictly less than before). Suppose for instance that φ is

$$R(t_0, \dots, t_{n-1})$$

and suppose $t_0 = F(s_0, \dots, s_k)$ where s_i are simpler terms. Then φ is equivalent to

$$\exists x_0, \dots, x_{n-1}, y\ (y{=}F(\bar{x}) \wedge \bigwedge(x_i = s_i) \wedge R(y, t_2, \dots, t_{n-1})).$$

φ is also equivalent to the formula

$$\forall x_0, \dots, x_{n-1}, y\ (y{=}F(\bar{x}) \wedge \bigwedge(x_i = s_i) \longrightarrow R(y, t_2, \dots, t_{n-1})).$$

then we complete the proof by induction. Of course one needs to do a similar reduction in the case that φ is an equality of terms, or a more general formula. □

Remark. The above procedure (as described in the proof of the lemma) is analogous to a procedure in the theory of differential equations. Here one can turn an order n differential equation in one variable into an equivalent first-order differential equation in n variables. For instance given the equation

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} f = 0.$$

Then by defining $\bar{y} = (f, \frac{d}{dt}f, \frac{d^2}{dt^2}f, \dots, \frac{d^{n-1}}{dt^{n-1}}f)$ we get an equivalent system of first-order differential equations

$$\sum_{i=0}^{n-1} a_i y_i + a_n \frac{d}{dt} y_{n-1} = 0$$

where

$$y_{i+1} = \frac{d}{dt} y_i.$$

Unnested formulae are useful when dealing with an interpretation of one language in another language where they allow us to deal with just the basic structure.

Definitional expansions

There are cases where, when extending the language in some sense gives no further structure, i.e. whatever new structure we get in the new language, was already there in the old language. An example will make this clear.

Example. Let $(\mathbb{R}, 0, 1, +, \cdot)$ be $\mathbb{R}$ in the signature of rings, $\tau = \{0, 1, +, \cdot\}$. In this structure the ordinary relation $x \leq y$ on $\mathbb{R}$ is already definable! For instance we could set

$$x \leq y \qquad \text{iff} \qquad \exists z\ (x + z^2 = y).$$

Now $\leq$ is not in the signature, but the *set* (in $\mathbb{R}^2$) given by the relation $x \leq y$ is definable in $(\mathbb{R}, 0, 1, +, \cdot)$. Thus extending the signature to the signature of ordered rings, i.e. $\tau^+ = \tau \cup \{\leq\}$ seems not to give us any new definable sets[2].

Definition. If $\tau \subseteq \tau'$ is an extension of signatures, and $\mathfrak{A}'$ is a τ'-structure and $\mathfrak{A} := \mathfrak{A}'|_\tau$ we say that $\mathfrak{A}'$ is a **definitional expansion of** $\mathfrak{A}$ if every τ'-definable set is already τ-definable.

This definition requires us to look at all τ'-definable sets. The following equivalent criterion allows us to focus on the definitions of the symbols of τ' in terms of the simpler language $\mathscr{L}(\tau)$.

Theorem 2. *Let $\tau \subseteq \tau'$ be an extension of signatures and let $\mathfrak{A} = \mathfrak{A}'|_\tau$. If*

- *for each $c \in \mathcal{C}_{\tau'}$ there is some $\theta_c(x) \in \mathscr{L}(\tau)$ such that*

$$\mathfrak{A} \models \theta_c(a) \qquad \textit{iff} \qquad a = c^{\mathfrak{A}'}$$

- *for each $f \in \mathcal{F}_{\tau'}$ of arity n there is some $\theta_f(\bar{x}, y)$ such that*

$$\mathfrak{A} \models \theta_f(\bar{a}, b) \qquad \textit{iff} \qquad f^{\mathfrak{A}'}(\bar{a}) = b$$

- *and for each $R \in \mathcal{R}_{\tau'}$ of arity n there is some $\theta_R(\bar{x})$ such that*

$$\mathfrak{A} \models \theta_R(\bar{a}) \qquad \textit{iff} \qquad \bar{a} \in R^{\mathfrak{A}'}.$$

Then $\mathfrak{A}'$ is a definitional expansion of $\mathfrak{A}$.

Proof. Let $\chi(\bar{x})$ be a formula in $\mathscr{L}(\tau')$. We want a formula $\tilde{\chi}(\bar{x})$ in $\mathscr{L}(\tau)$ such that $\tilde{\chi}(\mathfrak{A}') = \chi(\mathfrak{A}')$. Since every formula is equivalent to an unnested formula we may assume that χ is unnested. Now we work by induction on the complexity of χ.

- The case where χ is atomic is covered immediately by the assumptions of the theorem.
- If χ is $\chi_1 \wedge \chi_2$ then by induction hypothesis χ_1 and χ_2 have equivalent forms, and so $\tilde{\chi} = \tilde{\chi}_1 \wedge \tilde{\chi}_2$. Similarly for disjunctions and negations.
- If χ is $\exists x\ \xi$ then $\tilde{\chi}$ is just $\exists x\ \tilde{\xi}$. In this last step we must be careful not to reuse variables.

[2] This is of course not a rigorous statement since there could *a priori* be some strange new definable sets when we introduce $\leq$ into the signature.

By induction we are done. $\square$

Remark. The fact that we can assume that χ is unnested makes the above proof much easier since we do not need to look carefully at all possible nested terms that occur in the formulas.

Example. From the theorem it is now clear that $(\mathbb{R}, 0, 1, +, \cdot, \leq)$ is in fact a definable expansion of $(\mathbb{R}, 0, 1, +, \cdot)$.