Math 225A – Model Theory

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General Information

These notes are based on a course in *Metamathematics* taught by Professor Thomas Scanlon at UC Berkeley in the Autumn of 2013. The course will focus on Model Theory and the course book is Hodges' *a shorter model theory*.

As with any such notes, these may contain errors and typos. I take full responsibility for such occurences. If you find any errors or typos (no matter how trivial!) please let me know at mps@berkeley.edu.

Lecture 14

Interpretations

Definition. An interpretation Γ of the ρ-structure $\mathfrak{B}$ in the τ-structure $\mathfrak{A}$ is given by

- a τ-formula $\partial_\Gamma(x_0, \ldots, x_{l-1})$
- for each unnested ρ-atomic formula $\varphi(y_0, \ldots, y_{m-1})$ a τ-formula

$$\varphi_\Gamma(x_{0,0}, \ldots, x_{l-1,0}; x_{0,1}, \ldots, x_{l-1,1}; \ldots\ldots x_{0,m-1}, \ldots, x_{l-1,m-1})$$

- and a surjective function
$$\pi : \partial_\Gamma(\mathfrak{A}) \longrightarrow \mathfrak{B}$$
such that for all $a, b \in \partial_\Gamma(\mathfrak{A})$ then $\pi(a) = \pi(b)$ if and only if $\varphi_\Gamma(a, b)$ where φ is the ρ-atomic formula $y_0 = y_1$.

The condition on the map π is just that it pulls back the equality relation on $\mathfrak{B}$ to the interpretation (via Γ) of the equality relation on $\mathfrak{A}$.

We give a couple of examples.

Example. A classic example of an interpretation is that of the complex numbers in the reals. Here we interpret a complex number $z \in \mathbb{C}$ as a pair of real numbers (a, b) (which we think of as $z = a + ib$) with addition and multiplication defined appropriately.

Formally we interpret $(\mathbb{C}, +, \cdot, 0, 1)$ in $(\mathbb{R}, +, \cdot, 0, 1)$ as follows; Let $\partial_\Gamma(x_0, x_1)$ be any true statement for example $x_0 = x_0$. Thus $\partial_\Gamma(\mathbb{R}) = \mathbb{R}^2$. Here are some of the crucial interpretations of the unnested formulae

- $(y = 0)_\Gamma$ will be $(x_0 = 0 \wedge x_1 = 0)$.
- $(y = 1)_\Gamma$ will be $(x_0 = 1 \wedge x_1 = 0)$.
- $(y_2 = y_0 + y_1)_\Gamma$ will be $(x_{0,0} + x_{0,1} = x_{0,2} \wedge x_{1,0} + x_{1,1} = x_{1,2})$

- $(y_2 = y_0 \cdot y_1)_\Gamma$ will be $(x_{0,2} = x_{0,0}x_{0,1} - x_{1,0}x_{1,1} \wedge x_{1,0}x_{0,1} + x_{0,0}x_{1,1})$

finally the map $\pi : \partial_\Gamma(\mathbb{R}) \longrightarrow \mathbb{C}$ is given by $\pi(a_0, a_1) = a_0 + a_1\sqrt{-1}$. Here the usual equality relation in $\mathbb{C}$ pulls back to the coordinate-wise equality relation on $\partial_\Gamma(\mathbb{R}) = \mathbb{R}^2$ as it should!

Example. Set theory is stronger than arithmetic. I.e. we can also interpret arithmetic inside of set theory.

Let $\tau = \{\in\}$ be the signature of set theory and let $\mathbb{V}$ be a model of ZFC. Let ρ be the language of arithmetic, $\rho = \{\leq, +, \cdot, 0, 1\}$ and let $\mathfrak{B} = (\mathbb{N}, +, \cdot, 0, 1, \leq)$. We let $\partial_\Gamma(x)$ be the τ-formula which says "x is a natural number", this can be formally expressed in the language $\mathscr{L}(\tau)$ but we will not do so now. Now addition and multiplication can be given their usual set-theoretical interpretations (which again we will not properly write out).

Example. The example of $\mathbb{C}$ interpreted in $\mathbb{R}$ generalizes to any *finite* field extension. I.e. if L/K is a finite field extension then $(L, +, \cdot, 0, 1)$ is interpretable in $(K, +, \cdot, 0, 1)$.

In the definition of interpretation we only required there to be interpretations of unnested atomic formulae, but in fact there is a natural way to associate any ρ-formula to a τ-formula.

Proposition. *Given an interpretation Γ of $\mathfrak{B}$ in $\mathfrak{A}$ there is a natural function*

$$(-)_\Gamma : \mathscr{L}(\rho) \longrightarrow \mathscr{L}(\tau)$$

such that $\mathfrak{A} \models (\varphi)_\Gamma(\bar{a})$ if and only if all a_i satisfy ∂_Γ and $\mathfrak{B} \models \varphi(\pi\bar{a})$. The association is given inductively by

- *for φ atomic unnested $(\varphi)_\Gamma$ is φ_Γ (as given in the definition of an interpretation)*
- $(\varphi \wedge \psi)_\Gamma$ *is* $(\varphi)_\Gamma \wedge (\Psi)_\Gamma$
- $(\neg\varphi)_\Gamma$ *will be* $\neg(\varphi)_\Gamma \wedge \bigwedge \partial_\Gamma(-)$
- $(\exists x\varphi)_\Gamma$ *will be* $(\exists y_0, \ldots, y_{l-1})(\partial_\Gamma(\bar{y}) \wedge \varphi_\Gamma(\bar{y}))$

Proof. Immediate from the construction of Γ. □

Given a collection of formulae in $\mathscr{L}(\tau)$: ∂_Γ and φ_Γ for φ an unnested formula in $\mathscr{L}(\rho)$ then we want a theory T_Γ which says that these formulae give an interpretation. I.e. T_Γ asserts that for any $\mathfrak{A}$ which models T_Γ then the data ∂_Γ and φ_Γ define an interpretation. More precisely T_Γ must say

- If φ has n free variables and ∂_Γ has m free variables then φ_Γ has mn free variables.

- $(y_0 = y_1)_\Gamma$ is an equivalence relation $\sim$ on $\partial_\Gamma(-)$.
- for each $f \in \mathcal{F}_\rho$ if φ is $f(x) = y$ then T_Γ must say that

$$\forall \bar{u} \exists \bar{v} \varphi_\Gamma(\bar{u}, \bar{v}) \wedge \forall \bar{u}, \forall v, w(\varphi_\Gamma(\bar{u}, v) \wedge \varphi_\Gamma(\bar{u}, w) \longrightarrow v \sim w)$$

- For each constant $c \in \mathcal{C}_\rho$, if φ is $y = c$, then T_Γ must say that

$$\exists \bar{x} \varphi_\Gamma(\bar{x}) \wedge \forall \bar{x}, \bar{y} \varphi_\Gamma(\bar{x}) \wedge \varphi_\Gamma(\bar{y}) \longrightarrow x \sim y$$

  and

$$x \sim y \wedge \varphi_\Gamma(x) \longrightarrow \varphi_\Gamma(y)$$

- For $R \in \mathcal{R}_\rho$ then if $\varphi(x)$ is $R(x)$ we have that T_Γ must say that

$$\forall \bar{u}, \bar{v} \varphi_\Gamma(\bar{u}) \wedge \bar{u} \sim \bar{v} \longrightarrow \varphi_\Gamma(\bar{v}).$$

Proposition. *If* $\mathfrak{A} \models T_\Gamma$ *then* Γ *is an interpretation of* $\Gamma(\mathfrak{A}) := \mathfrak{B}$ *where* $\mathrm{dom}(\mathfrak{B}) := \partial_\Gamma(\mathfrak{A})/\sim$. *Here we have*

- *for constants c we have* $c^{\mathfrak{B}} := [\bar{b}]_\sim$ *for any* $\bar{a} \in \partial_\Gamma(\mathfrak{A})$ *such that* $\mathfrak{A} \models (x{=}c)_\Gamma(\bar{a})$.
- $([\bar{a}_0]_\sim, \ldots, [\bar{a}_{n-1}]_\sim) \in R^{\mathfrak{B}}$ *iff* $\mathfrak{A} \models (R(x))_\Gamma(\bar{a})$
- *and* $f^{\mathfrak{B}}([\bar{a}]_\sim) = [b]_\sim$ *iff* $\mathfrak{A} \models (f(\bar{x}) = y)_\Gamma(\bar{a}, b)$.

Proof. We defined T_Γ so as to say exactly what this proposition is saying. □

Example. If $\mathfrak{A}'$ is a definitional expansion on $\mathfrak{A}$ then the definitional expansion *is* an interpretation of $\mathfrak{A}'$ in $\mathfrak{A}$.

A useful observation (which we will now prove) is that an interpretation Γ preserves elementary substructures.

Proposition. *If* $(\partial_\Gamma, \{\varphi_\Gamma : \varphi$ *unnested ρ-formula*$\})$ *is given and* $\mathfrak{A} \preccurlyeq \mathfrak{A}'$ *where* $\mathfrak{A}' \models T_\Gamma$ *then* $\Gamma(\mathfrak{A}) \preccurlyeq \Gamma(\mathfrak{A}')$.

Proof. Since $\mathfrak{A} \preccurlyeq \mathfrak{A}'$ we have $\partial_\Gamma(\mathfrak{A}) \subseteq \partial_\Gamma(\mathfrak{A}')$. Also $\sim$ the equivalence relation (given by $(x = y)_\Gamma$) on ∂_Γ is an equivalence relation on both $\partial_\Gamma(\mathfrak{A})$ and on $\partial_\Gamma(\mathfrak{A}')$. Furthermore, again since $\mathfrak{A} \preccurlyeq A'$ the restriction of $\sim$ on $\partial_\Gamma(\mathfrak{A}')$ to $\partial_\Gamma(\mathfrak{A})$ is just the old $\sim$.

So the inclusion

$$\partial_\Gamma(\mathfrak{A}) \longrightarrow \partial_\Gamma(\mathfrak{A}')$$

induces an inclusion

$$\partial_\Gamma(\mathfrak{A})/\sim^{\mathfrak{A}} \longrightarrow \partial_\Gamma(\mathfrak{A}')/\sim^{\mathfrak{A}'}.$$

The rest of the proof now follows from the earlier proposition: For any unnested formula φ in $\mathscr{L}(\rho)$ and tuple $\bar{a}$ from $\partial_\Gamma(\mathfrak{A})$ we have

$$\mathfrak{A} \models (\varphi)_\Gamma(\bar{a}) \qquad \text{iff} \qquad \Gamma(\mathfrak{A}) \models \varphi([\bar{a}]_\sim)$$

by the proposition. But by elementary extension we have

$$\mathfrak{A} \models (\varphi)_\Gamma(\bar{a}) \qquad \text{iff} \qquad \mathfrak{A}' \models (\varphi)_\Gamma(\bar{a})$$

and so again by the proposition we have

$$\mathfrak{A}' \models (\varphi)_\Gamma(\bar{a}) \qquad \text{iff} \qquad \Gamma(\mathfrak{A}') \models \varphi([\bar{a}]_\sim)$$

so $\Gamma(\mathfrak{A}) \preccurlyeq \Gamma(\mathfrak{A}')$. □

If one can interpret a class of ρ structures in some other class of τ-structures, then one can pass elementary embedding from one class to the other.

Interpretations induce continuous homomorphisms between automorphism groups. To prove this we first need a general lemma about topological groups.

Lemma. *Let G and H be topological groups and $\alpha : G \longrightarrow H$ a homomorphism. Then α is continuous if and only if α is continuous at the identity.*

Proof. The forward direction is clear.

Suppose α is continuous at the identity $1_G \in G$. Let $g \in G$ and let $U \subseteq H$ be an open subset containing $\alpha(g)$. Then translating U by $\alpha(g)^{-1}$ we see that $1_H \in \alpha(g)^{-1}U$. Now $\alpha(g)^{-1}U$ is also open since translation is a homeomorphism $H \to H$. Now by assumption there is some V open in G such that $1_G \in V$ and $\alpha(V) \subseteq \alpha(g)^{-1}U$. Thus gV contains g (and is open) and $\alpha(gV) \subseteq U$. □

Proposition. *To an interpretation Γ of $\mathfrak{B}$ in $\mathfrak{A}$ there is an associated continuous homomorphism*

$$\Gamma : \mathrm{Aut}(\mathfrak{A}) \longrightarrow \mathrm{Aut}(\mathfrak{B})$$

Proof. We first define the homomorphism.

Let σ be an automorphism of $\mathfrak{A}$. First note that σ must preserve $\partial_\Gamma(\mathfrak{A})$. I.e. $\mathfrak{A} \models \partial_\Gamma(\bar{a})$ if and only if $\mathfrak{A} \models \partial_\Gamma(\sigma\bar{a})$.

Now the equivalence relation $\sim$ is also defined by some formula, so σ also preserves this. I.e. $a \sim b$ iff $\sigma a \sim \sigma b$.

Thus σ induces a function, $\hat{\sigma}$ of equivalence classes $\partial(\mathfrak{A})/\sim$. Now the by the isomorphism $(\partial(\mathfrak{A})/\sim) \cong \mathfrak{B}$ we get a (bijective) function $\Gamma(\sigma) : \mathfrak{B} \to \mathfrak{B}$.

We must check that it is also an automorphism. It suffices to check that $\Gamma(\sigma)$ preserves unnested ρ-formulae. Let φ be an unnested ρ-formula and $\mathfrak{B} \models \varphi(\bar{b})$. This is equivalent to $\mathfrak{A} \models (\varphi)_\Gamma(\bar{a})$ (where $\bar{a} = \pi(\bar{b})$) which is equivalent to $\mathfrak{A} \models (\varphi)\Gamma(\sigma\bar{a})$ and finally this is equivalent to $\mathfrak{B} \models \varphi(\Gamma(\sigma)(\bar{b})$.

Finally we must also check the continuity of $\Gamma : \mathrm{Aut}(\mathfrak{A}) \longrightarrow \mathrm{Aut}(\mathfrak{B})$. For this we use the lemma: It suffices to check continuity at the identity. Let U be open subset of $\mathrm{Aut}(\mathfrak{B})$ containing $\Gamma(id_\mathfrak{A})$. Without loss of generality we may assume that

U is a basic open set around $id_{\mathfrak{B}}$, i.e. take U to be the stabilizer of $\bar{b}$ for some $\bar{b}$ from $\mathfrak{B}$. Let $\bar{a}$ be a finite tuple of $\mathfrak{A}$ such that $\bar{b} = \pi\bar{a}$ (which is possible since π is surjective). Then $\Gamma(\sigma)(U_{\bar{a},\bar{a}}) \subseteq U_{\bar{b},\bar{b}}$. So Γ is continuous. $\square$

Question. Suppose that Γ is an interpretation of $\mathfrak{B}$ in $\mathfrak{A}$ and Δ is an interpretation of $\mathfrak{A}$ in $\mathfrak{B}$. Must $\Delta \circ \Gamma : \mathrm{Aut}(\mathfrak{A}) \longrightarrow \mathrm{Aut}(\mathfrak{A})$ be an automorphism?

These and many other related questions have been heavily studied, see for example [1] and [2].

Bibliography

[1] Matatyahu Rubin, *The Reconstruction of Trees from Their Automorphism Groups.* American Mathematical Society, 1991.

[2] Peter J. Cameron, *Oligomorphic Permutation Groups.* Cambridge University Press, 1990.