Math 225A – Model Theory

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General Information

These notes are based on a course in *Metamathematics* taught by Professor Thomas Scanlon at UC Berkeley in the Autumn of 2013. The course will focus on Model Theory and the course book is Hodges' *a shorter model theory.*

As with any such notes, these may contain errors and typos. I take full responsibility for such occurences. If you find any errors or typos (no matter how trivial!) please let me know at mps@berkeley.edu.

Lecture 21

Interpolation

Theorem 1. *Given signatures $\tau_1, \tau_2 \supseteq \sigma$ such that $\tau_1 \cap \tau_2 = \sigma$, and given $\mathfrak{A}_1 \in \mathrm{Str}(\tau_1)$ and $\mathfrak{A}_2 \in \mathrm{Str}(\tau_2)$ such that $\mathfrak{A}_1|_\sigma \equiv \mathfrak{A}_2|_\sigma$, then there exists $\mathfrak{B}$ a $\tau_1 \cup \tau_2$-structure such that $\mathfrak{A}_1 \preccurlyeq \mathfrak{B}|_{\tau_1}$ and $\mathfrak{A}_2 \preccurlyeq \mathfrak{B}|_{\tau_2}$.*

Proof. Consider the theory

$$\mathrm{eldiag}(\mathfrak{A}_1) \cup \mathrm{eldiag}(\mathfrak{A}_2).$$

A model of this theory would suffice. If no such model exists then by compactness there are $\varphi(a) \in \mathrm{eldiag}(\mathfrak{A}_1)$ and $\psi(b) \in \mathrm{eldiag}(\mathfrak{A}_2)$ where a and b are new constants, $\psi, \varphi \in \mathscr{L}(\sigma)$ and

$$\vdash \varphi(a) \longrightarrow \neg\psi(b).$$

Thus any expansion of $\mathfrak{A}_1$ to an $\mathscr{L}(\tau_{1,b})$-structure must satisfy $\neg\psi(b)$, so $\mathfrak{A}_1 \models \forall x \neg\psi(x)$. Now $\forall x \neg\psi(x) \in \mathscr{L}(\sigma)$ and so $\mathfrak{A}_1|_\sigma \models \forall x \neg\psi(x)$. But since $\mathfrak{A}_1|_\sigma \equiv \mathfrak{A}_2|_\sigma$ we must have $\mathfrak{A}_2|_\sigma \models \forall x \neg\psi(x)$, contradicting the fact that $\psi(b) \in \mathrm{eldiag}(\mathfrak{A}_2)$. $\square$

From this theorem we get two syntactic consequences.

Notation. For T a τ-theory and $\sigma \subseteq \tau$ we denote by T_σ, the set of all σ-consequences of T, i.e. $\{\psi \in \mathscr{L}(\sigma) : \mathrm{T} \vdash \psi\}$.

Corollary. *If $\sigma \subseteq \tau$ is an extension of signatures and T is a τ-theory, then a σ-structure $\mathfrak{A}$ satisfies T_σ if and only if there is a model $\mathfrak{B}$ of T such that $\mathfrak{A} \preccurlyeq \mathfrak{B}|_\sigma$.*

Proof. Let $\mathfrak{A} \models \mathrm{T}_\sigma$. Consider the theory

$$\mathrm{T} \ \cup \ \mathrm{eldiag}(\mathfrak{A})$$

(remember that $\mathrm{eldiag}(\mathfrak{A})$ is a σ-theory). If this were a consistent theory the we would be done. If not then, by compactness, there is some $\psi(a) \in \mathrm{eldiag}(\mathfrak{A})$ such that $\mathrm{T} \cup \{\psi(a)\}$ is inconsistent. Here $\psi \in \mathscr{L}(\sigma)$ and a is a tuple of new constants. So we have

$$\mathrm{T} \vdash \forall x \neg\psi(x)$$

i.e. $\forall x \neg\psi(x) \in \mathrm{T}_\sigma$, contradicting the fact that $\mathfrak{A} \models \mathrm{T}_\sigma$. Thus we get the desired model.

The converse implication is clear. $\square$

Remark. Note that the Corollary does *not* claim that $\mathfrak{A}$ is a reduct of a model of T. To see that this is false in general, consider $\sigma = \{<\}$ and $\tau = \{<, +, 0\}$ and T the theory of divisible ordered abelian groups. Then T_σ is the theory of dense linear orders without endpoints. Then $\mathbb{Q} \oplus \mathbb{R} \models \mathrm{T}_\sigma$, but there is no way to order $\mathbb{Q} \oplus \mathbb{R}$ to make it satisfy T (since it is not homogeneous).

Corollary. *(Interpolation Theorem) Given $\tau_1, \tau_2 \supseteq \sigma$ with $\tau_1 \cap \tau_2 = \sigma$ and $\mathrm{T}_1, \mathrm{T}_2$ theories in $\mathscr{L}(\tau_1)$, $\mathscr{L}(\tau_2)$ respectively. If $\mathrm{T}_1 \cup \mathrm{T}_2$ is inconsistent, then there is a sentence $\psi \in \mathscr{L}(\sigma)$ such that $\mathrm{T}_1 \vdash \psi$ and $\mathrm{T}_2 \vdash \neg\psi$.*

Proof. Consider the theory $(\mathrm{T}_1)_\sigma \cup (\mathrm{T}_2)_\sigma$. If this is inconsistent then we're done. If it is consistent then let $\mathfrak{A}$ be a model. Note that $\mathfrak{A}$ is a σ-structure. By the Corollary there exists a model $\mathfrak{B}_1 \models \mathrm{T}_1$ such that $\mathfrak{A} \preccurlyeq \mathfrak{B}_1|_\sigma$, and a model $\mathfrak{B}_2 \models \mathrm{T}_2$ such that $\mathfrak{A} \preccurlyeq \mathfrak{B}_2|_\sigma$. But then $\mathfrak{B}_1|_\sigma \equiv \mathfrak{B}_2|_\sigma$ and so by the Theorem, there exists some $\tau_1 \cup \tau_2$-structure $\mathfrak{C}$ such that $\mathfrak{B}_1 \preccurlyeq \mathfrak{C}|_{\tau_1}$ and $\mathfrak{B}_2 \preccurlyeq \mathfrak{C}|_{\tau_2}$. But then $\mathfrak{C} \models \mathrm{T}_1 \cup \mathrm{T}_2$ contrary to our assumption. $\square$

As a Corollary we get the Craig Interpolation Theorem.

Theorem 2. *(Craig Interpolation) Given τ_1, τ_2 and σ as above, and $\varphi \in \mathscr{L}(\tau_1)$, $\psi \in \mathscr{L}(\tau_1)$. If $\varphi \vdash \psi$ then there exists $\theta \in \mathscr{L}(\sigma)$ such that $\varphi \vdash \theta$ and $\theta \vdash \psi$.*

One other consequence of the theorem is called Beth's Definability Theorem. It states roughly that explicit and implicit definability are equivalent for first-order logic.

Theorem 3. *(Beth's Definability Theorem) Let $\sigma \subseteq \tau$ be an extension of signatures, T a τ-theory and $\varphi(x) \in \mathscr{L}(\tau)$. Then the following are equivalent.*

1) *(Implicit) For all models $\mathfrak{A}, \mathfrak{B} \models \mathrm{T}$, if $\mathfrak{A}|_\sigma = \mathfrak{B}|_\sigma$ then $\varphi(\mathfrak{A}) = \varphi(\mathfrak{B})$.*
2) *(Explicit) There is some $\psi \in \mathscr{L}(\sigma)$ such that $\mathrm{T} \vdash \varphi \leftrightarrow \psi$.*

Proof. 2) $\Longrightarrow$ 1). Clear.

1) $\Longrightarrow$ 2). Consider the theory (in the extended language $\mathscr{L}(\tau_{a,b})$)

$$S := \mathrm{T} \ \cup\ \{\varphi(a)\} \ \cup\ \{\neg\varphi(b)\} \ \cup\ \{\psi(a) \leftrightarrow \psi(b) : \psi \in \mathscr{L}(\sigma)\}.$$

If S is inconsistent then we're done, since by compactness there would be a finite set Ψ of $\mathscr{L}(\sigma)$-formulae such that

$$\mathrm{T} \vdash \forall x, y \left[\bigwedge_{\psi \in \Psi} (\psi(x) \leftrightarrow \psi(y)) \longrightarrow (\varphi(x) \leftrightarrow \varphi(y)) \right].$$

Now set

$$\theta := \bigvee_{\substack{\mu \subseteq \Psi \text{ s.t.} \\ T \cup \{\varphi(x)\} \cup \bigwedge \mu \cup \bigwedge_{\psi \notin \mu} \neg\psi \\ \text{is consistent}}} \left(\bigwedge_{\psi \in \mu} \psi \wedge \bigwedge_{\psi \notin \mu} \neg\psi \right)$$

Then $\mathrm{T} \vdash \varphi \leftrightarrow \theta$.

So suppose S is consistent. Let $(\mathfrak{C}, a, b)$ be a model of S. We will now apply Theorem 1. We define two new signatures. Let τ_1 and τ_2 be disjoint copies (over σ) of τ together with a new constant symbol c. More precisely we can decorate each symbol of $\tau \setminus \sigma$ with a subscript either 1 or 2. Thus τ_1 consists of symbols from σ together with symbols $x^{(1)}$ for all $x \in \tau \setminus \sigma$ and also contains the new symbol c. Similarly for τ_2. So $\tau_1, \tau_2 \supseteq \sigma_c$ and $\tau_1 \cap \tau_2 = \sigma_c$.

Let $\mathfrak{M}$ be $(\mathfrak{C}, a)$ considered as a τ_1-structure, i.e. $c^{\mathfrak{M}} = a$ and $x^{(1)} = x^{\mathfrak{C}}$. Similarly let $\mathfrak{N}$ be $(\mathfrak{C}, b)$ considered as a τ_2-structure.

Now since we arranged that a and b have the same σ-type (since $(\mathfrak{C}, a, b) \models S$) we have that

$$\mathfrak{M}|_{\sigma_c} \equiv \mathfrak{N}|_{\sigma_c} \qquad \text{i.e. } (\mathfrak{C}|_\sigma, a) \equiv (\mathfrak{C}|_\sigma, b).$$

Now by Theorem 1 there exists $\mathfrak{D}$ a $(\tau_1 \cup \tau_2)$-structure such that $\iota_1 : \mathfrak{M} \preccurlyeq \mathfrak{D}|_{\tau_1}$ and $\iota_2 : \mathfrak{N} \preccurlyeq \mathfrak{D}|_{\tau_2}$ (elementary embeddings). Note that $\iota_1(a) = \iota_2(b)$ since $c^{\mathfrak{D}} = a$ and $c^{\mathfrak{D}} = b$.

Let $\mathfrak{A}$ be $\mathfrak{D}|_{\tau_1}$ regarded as a τ-structures (i.e. forgetting the constant c). Similarly let $\mathfrak{B}$ be $\mathfrak{D}|_{\tau_2}$ regarded as a τ-structure. Then we have a literal equality

$$\mathfrak{A}|_\sigma = \mathfrak{B}|_\sigma$$

since then are both equal to $\mathfrak{D}|_\sigma$. Now $\mathfrak{A}$ and $\mathfrak{B}$ both model T since $\mathfrak{C}$ was a model of T. But they disagree on φ, i.e. $\varphi(\mathfrak{A}) \neq \varphi(\mathfrak{B})$ since $\mathfrak{A} \models \varphi(c^{\mathfrak{D}})$ and $\mathfrak{B} \models \neg\varphi(c^{\mathfrak{D}})$. This contradicts the assumption (1), thus S must be inconsistent. This completes the proof. $\square$

Indiscernibles

Indiscernibles are a tool for analyzing structures by making them much more homogeneous. By making them more homogeneous we can take local information and expand it to get global information about the structures.

Definition. A sequence $(a_i)_{i\in\omega}$ in some τ-structure $\mathfrak{A}$ is an **indiscernible sequence** if for any formula $\varphi(x_0,\dots,x_{n-1})$ and $i_0<\dots<i_{n-1}$ and $j_0<\dots<j_{n-1}$ increasing sequences from ω then

$$\mathfrak{A}\models\varphi(a_{i_0},\dots,a_{i_{n-1}})\longleftrightarrow\varphi(a_{j_0},\dots,a_{j_{n-1}}).$$

Remark. If $(a_i)_{i\in\omega}$ is an indiscernible sequences then the type of an increasing n-sequence $a_{i_0},\dots,a_{i_{n-1}}$ is constant, i.e. is the same for all such increasing n-sequences from $(a_i)_{i\in\omega}$.

In particular any two elements a_i and a_j from the sequence have the same type.

Definition. If the order does not matter then the sequence $(a_i)_{i\in\omega}$ is called an **indiscernible set**. More precisely the requirement is that for any set $J\subseteq\omega$ of size n, say $J=\{j_0,\dots,j_{n-1}\}$ then

$$\mathfrak{A}\models\varphi(a_0,\dots,a_{n-1})\longleftrightarrow\varphi(a_{j_0},\dots,a_{j_{n-1}}).$$

Of course an indiscernible set is in particular an indiscernible sequence.

Example. If $a_0<a_1<\dots\in\mathbb{Q}$ then $(a_i)_{i\in\omega}$ is an indiscernible sequence in $\mathbb{Q}$ considered as an ordered structure. It is not an indiscernible *set*.

Example. If X is any infinite set in the language of equality (i.e. $\tau=\emptyset$) and $(a_i)_{i\in\omega}$ is any sequence without repititions from X then it is an indiscernible set. Alternatively if $(a_i)_{i\in\omega}$ is the constant sequence then it is also an indiscernible set.

Example. Let V be a vector space over a field k in the language of vector spaces $\tau=\{+,(\lambda)_{\lambda\in k}\}$. Then any linearly independent set $X\subseteq V$ is an indiscernible set. To see this note that we can extend X to a basis for V, and that a change of bases extends to an automorphism of V.

Our goal is to show the following:

Proposition. *If* T *is any* τ*-theory and* $\Sigma(x)$ *a set of* $\mathscr{L}(\tau_x)$*-formulae such that it is consistent that there exists a model* $\mathfrak{A}$ *of* T *such that* $\Sigma(\mathfrak{A})$ *is infinite, then there exists a model* $\mathfrak{B}$ *of* T *and a sequence* $(a_i)_{i\in\omega}$ *which is non-constant and is an indiscernible sequence such that* $\mathfrak{B}\models\Sigma(a_i)$ *for all* $i\in\omega$.

We shall begin the proof, but we will need Ramsey's theorem at some point. The proof of Ramsey's theorem will be given afterwards.

Proof. (Assuming Ramsey's Theorem) We write down what we want: Let S be the theory

$$\begin{aligned}&\mathrm{T}\ \cup\ \bigcup_{i=0}^{\infty}\Sigma(x_i)\ \cup\ \{x_i\neq x_j\mid i\neq j\}\\&\cup\{\psi(x_{i_0},\dots,x_{i_{n-1}})\leftrightarrow\psi(x_{j_0},\dots,x_{j_{n-1}})\mid\psi\in\mathscr{L}(\tau),i_0<\dots<i_{n-1}\text{ and }j_0<\dots<j_{n-1}\}\end{aligned}$$

If S is consistent then we are done since the interpretations of the x_i's would be a non-constant indiscernible sequence.

Suppose therefore that S is not consistent. Then by compactness there is some finite fragment which is inconsistent. Then there is some $N \in \mathbb{N}$ such that the theory

$$\begin{aligned}
&\mathrm{T} \;\cup\; \{\theta(x_i) \mid i \leq N\} \;\cup\; \{x_i \neq x_j \mid i \neq j \leq N\} \\
&\cup \{\psi_k(x_{i_0}, \ldots, x_{i_{n_k-1}}) \leftrightarrow \psi_k(x_{j_0}, \ldots, x_{j_{n_k-1}}) \mid \psi_k \in \mathscr{L}(\tau), k \leq K, \\
&i_0 < \cdots < i_{n-1} \leq N \text{ and } j_0 < \cdots < j_{n-1} \leq N\}
\end{aligned}$$

is inconsistent. We may assume (by way of padding) that there is some n such that $n_k = n$ for all k.

Now we know that there is some model $\mathfrak{A}$ of T such that θ has infinitely many realizations, i.e. $|\theta(\mathfrak{A})| \geq \aleph_0$. Let $b_0, b_1, \ldots$ be a sequence of distinct elements from $\theta(\mathfrak{A})$.

Notation. The set $[\omega]^n$ consists of all strictly increasing n-tuples. I.e. $[\omega]^n := \{(l_1, \ldots, l_n) \in \omega^n : l_1 < l_2 < \cdots < l_n\}$.

Define a function

$$f : [\omega]^n \longrightarrow \mathcal{P}(\{1, \ldots, k\})$$

by

$$f(i_0, \ldots, i_{n-1}) := \{k \mid \mathfrak{A} \models \psi_k(b_{i_0}, \ldots, b_{i_{n-1}})\}$$

Now f is a function from $[\omega]^n$ to a finite set. By Ramsey's theorem (see below for statement and proof) there exists $H \subseteq \omega$ infinite and homogeneous, i.e. $f|_{[H]^n}$ is constant. Let $H = \{a_0 < a_1 < \ldots\}$. Interpret x_i in $\mathfrak{A}$ as a_i. This will satisfy out purportedly inconsistent sub theory. This yields a contradiction and completes the proof (modulo Ramsey's theorem). □

We need fill the gap in the above proof.

Notation. In the course of the proof we introduced the notation $[\omega]^n$ for all increasing n-sequences from ω.

Theorem 4. *(Ramsey's Theorem) Given a function f from $[\omega]^n$ (for some $n \in \omega$) to a finite set, then there exists an infinite subset H of ω such that f is constant $[H]^n$*

Proof. We may assume that the codomain of f is in fact $\{0, \ldots, N-1\}$ (where N is to the cardinality of the codomain).

Consider the structure $\mathfrak{A} = (\omega, <, \{k\}_{k \in \omega}, f)$, where $<$ is interpreted as the standard order on ω and $f^{\mathfrak{A}}$ is interpreted to be the same as the given function f expect that $f^{\mathfrak{A}}(b_0, \ldots, b_{n-1}) = 0$ if $(b_0, \ldots, b_{n-1}) \notin [\omega]^n$.

We will prove the theorem by induction on n.

For $n = 1$ the theorem follows from the pigeon hole principle.

For $n + 1$, suppose the theorem holds for all integers $\leq n$.

Take a *proper* elementary extension $\mathfrak{A}^*$ of $\mathfrak{A}$, which is possible by upward Löwenheim-Skolem. In particular $\mathfrak{A} \equiv \mathfrak{A}^*$. So $(\mathfrak{A}^*, <)$ is a linear order. Let $a \in \mathrm{dom}(\mathfrak{A}^*) \setminus \omega$ be a new element from $\mathfrak{A}^*$. Note that $a > n$ for every $n \in \omega$ since for all $n \in \omega$ the structure $\mathfrak{A}$ satisfies that n has exactly n predecessors, hence $\mathfrak{A}^*$ must satisfy this as well. But then we cannot have $a \leq n$ for any $n \in \omega$ and so by the linearity of the order we must have $a > n$. So a is an "infinite" number in $\mathfrak{A}^*$.

We construct an increasing sequence $m_0 < m_1 < \dots$ from ω. The first n elements are not important we just pick them such that $m_0 < m_1 < \cdots < m_{n-1}$. Now with $m_0 < \cdots < m_{j-1}$ constructed we search for an element $x > m_{j-1}$ such that

- for each $i_0 < \cdots < i_{n-1} \leq j - 1$ we have

$$f^{\mathfrak{A}^*}(m_{i_0}, \dots, m_{i_{n-1}}, a) = f^{\mathfrak{A}^*}(m_{i_0}, \dots, m_{i_{n-1}}, x).$$

I.e. x must behave like a with respect to the sequence $m_{i_0}, \dots, m_{i_{n-1}}$. Such an x will then be the j'th element of the sequence $m_0 < m_1 < \dots$. This puts finitely many constraints on x and so we can write it out as a first-order formula.

Consider the formula $\theta(x)$ given by

$$x{>}m_{j-1} \ \wedge \bigwedge_{i_0<\cdots<i_{n-1}\leq j-1} f(m_{i_0}, \dots, m_{i_{n-1}}, x) = f^{\mathfrak{A}^*}(m_{i_0}, \dots, m_{i_{n-1}}, a)$$

[Note: the first instance of the symbol f in θ is just a symbol, the second instance "$f^{\mathfrak{A}^*}(...)$" is the actual value of $f^{\mathfrak{A}^*}$ on the tuple $(m_{i_0}, \dots, m_{i_{n-1}}, a)$, i.e. a number in $\{0, \dots, N - 1\}$.]

We we have $\mathfrak{A}^* \models \theta(a)$ and so $\mathfrak{A}^* \models \exists x\theta(x)$. Now since $\mathfrak{A} \preccurlyeq \mathfrak{A}^*$ we have $\mathfrak{A} \models \exists x\theta(x)$. So let m_j be a witness, then we have the next element of the sequence: $m_0 < \cdots < m_{j-1} < m_j$.

Now we use the induction hypothesis: Define $g : [\omega]^n \longrightarrow \{0, \dots, N\}$ by

$$g(l_1, \dots, l_n) = f^{\mathfrak{A}^*}(m_{l_1}, \dots, m_{l_n}, a).$$

By the induction hypothesis there exists a homogenous set H such that g is constant on $[H]^n$. Now we claim that f is constant on the subset of H given by the l sequence m_l, i.e.

$$f|_{[\{m_l:l\in H\}]^{n+1}} \text{ is constant}$$

To see this, suppose $i_0 < \cdots < i_n$ and $j_0 < \cdots < j_n$ then

$$\begin{aligned}
f(m_{l_{i_0}}, \ldots, m_{l_{i_n}}) &= f^{\mathfrak{A}^*}(m_{l_{i_0}}, \ldots, m_{l_{i_{n-1}}}, a) \\
&= g(l_{i_0}, \ldots, l_{n-1}) \\
&= g(l_{j_0}, \ldots, l_{j_{n-1}}) \\
&= f^{\mathfrak{A}^*}(m_{l_{j_0}}, \ldots, m_{l_{j_{n-1}}}, a) \\
&= f(m_{l_{j_0}}, \ldots, m_{l_{j_n}}).
\end{aligned}$$

The theorem follows by induction. $\square$